INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 13, Number 4, Pages 610–626

A SPLITTING LEAST-SQUARES MIXED FINITE ELEMENT METHOD FOR ELLIPTIC OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper, we propose a splitting least-squares mixed finite element method for the approximation of elliptic optimal control problem with the control constrained by pointwise inequality. By selecting a properly least-squares minimization functional, we derive equivalent two independent, symmetric and positive definite weak formulation for the primal state variable and its flux. Then, using the first order necessary and also sufficient optimality condition, we deduce another two corresponding adjoint state equations, which are both independent, symmetric and positive definite. Also, a variational inequality for the control variable is involved. For the discretization of the state and adjoint state equations, either RT mixed finite element or standard C^0 finite element can be used, which is not necessary subject to the Ladyzhenkaya-Babuska-Brezzi condition. Optimal a priori error estimates in corresponding norms are derived for the control, the states and adjoint states, respectively. Finally, we use some numerical examples to validate the theoretical analysis.

Key words. Optimal control, splitting least-squares, mixed finite element method, positive definite, a priori error estimates.

1. Introduction

Optimal control problems are playing an increasingly important role in modern scientific and engineering numerical simulations. Nowadays, finite element method seems to be the most widely used numerical method in practical computation. The readers are referred to, for example, Refs. [1, 2, 3, 4, 5] for systematic introductions of finite element methods and optimal control problems.

In this paper, we are interested in the following convex quadratic optimal control problem with the control constrained by pointwise inequality:

$$\min_{u\in U_{ad}} \mathcal{J}(y,\sigma,u) = \frac{1}{2}\left(\int_\Omega (y-y_d)^2 + \int_\Omega (\sigma-\sigma_d)^2 + \gamma\int_{\Omega_U} u^2\right) \tag{1}$$

subject to

$$\begin{cases} \mathrm{div}\sigma + cy = f + \mathcal{B}u, & \text{in } \Omega, \\ \sigma + \mathcal{A}\nabla y = 0, & \text{in } \Omega, \\ y = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

and

$$\xi_1 \le u(x) \le \xi_2,\ a.e. \text{ in } \Omega_U. \tag{3}$$

Here $\gamma > 0$ is a constant, Ω and $\Omega_U \subseteq \Omega$ are two bounded domain in $\mathbb{R}^2$, with Lipschitz boundaries $\partial\Omega$ and $\partial\Omega_U$. A precise formulation of this problem including a functional analytic setting is given in the next section.

Received by the editors June 22, 2015 and, in revised form, December 3, 2015.

2000 *Mathematics Subject Classification.* 49K20, 49M25, 65N15, 65N30.

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This research was supported by the NSF of China (11201485, 91330106, 11571367, 11401588), the NSF of Shandong Province (BS2013NJ001), and the Fundamental Research Funds for the Central Universities (14CX02217, 15CX08004A).

Classical mixed finite element methods (see Refs. [6, 7, 8, 9, 10]) have been proved effectively for solving elliptic equations and fluid problems. They have an advantage of approximating the unknown scalar variable and its diffusive flux simultaneously. Besides, these methods can approximate the unknown variable and its flux to a same order of accuracy. Recently, there are some research articles on these methods for solving optimal control problems, see Refs. [11, 12, 13], for example. However, it is well known that these methods usually produce a symmetric but indefinite system for elliptic equations. Thus the popular conjugate gradient (CG) or algebraic multi-grid (AMG) solvers can not be used for the solution of linear algebraic equation systems.

To conquer these difficulties appeared in using classical mixed finite element methods, least-squares mixed finite element method, for first-order elliptic mixed system in unknown variable y and unknown velocity flux σ, was introduced by Pehlivanov et al. [14]. It is well known that the least-squares mixed finite element method has two typical advantages: First, it is not subjected to the Ladyzhenkaya-Babuska-Brezzi consistency condition, so the choice of finite element spaces becomes flexible; Second, it results in a symmetric and positive definite system, which can be solved using those solvers such as CG and AMG quickly. The idea of splitting least-squares was first proposed by Rui et al. in [15] for a reaction-diffusion equation, where by selecting a properly least-squares functional, the authors derived two independent, symmetric and positive definite equations, respectively, for the unknown state variable y and its flux σ. Then it is applied to solve linear and nonlinear parabolic equations [16], sobolev equations [17], pseudo-parabolic equations [18], and nonlinear convection-diffusion equations [19] and so on.

In this paper, we apply the splitting least-squares mixed finite element method for the discretization of elliptic optimal control problem. Pointwise inequality constraints on the control variable are considered. We derive optimal a priori error estimates, respectively, for the optimal control u^* in $L^2(\Omega_U)$-norm, which is approximated by piecewise constant or piecewise linear discontinuous elements; for the primal state y^* and adjoint state z^* both in $L^2(\Omega)$-norm and $H^1(\Omega)$-norm, which are approximated by standard piecewise linear C^0 finite elements; for the flux state σ^* and adjoint state ω^* in $H(\mathrm{div};\Omega)$-norm, which are approximated by the lowewt-order RT mixed finite elements or standard piecewise linear C^0 finite elements. Here, the Ladyzhenkaya-Babuska-Brezzi consistency condition for the discretization spaces of y^* and σ^* is not needed.

This paper is organized as follows. In Sect. 2, we introduce the optimal control problem and derive the continuous optimality conditions based on the idea of least-squares. In Sect. 3, a splitting least-squares mixed finite element approximation to the continuous optimal control problem is proposed, and then we derive the corresponding discrete optimality conditions. In Sect. 4, some a priori error estimates for the states, adjoint states and control are derived under control constrained by pointwise inequality. In Sect. 5, we conduct some numerical experiments to observe the convergence behavior of the numerical scheme. In the last section, some concluding remarks are given.

In the following, we employ the standard notations $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{W^{m,p}(\Omega)}$ and seminorm $|\cdot|_{W^{m,p}(\Omega)}$. We set $W_0^{m,p}(\Omega)=\{v\in W^{m,p}(\Omega): v=0 \text{ on } \partial\Omega\}$. For $p=2$, we denote $H^m(\Omega)=W^{m,2}(\Omega)$ and $H_0^m(\Omega)=W_0^{m,2}(\Omega)$. In addition, C denotes a general positive constant which is independent of the spatial mesh parameters.