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FINITE DIFFERENCE SCHEMES FOR THE KORTEWEG-DE VRIES-KAWAHARA EQUATION

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Abstract. We are concerned with the convergence of fully discrete finite difference schemes for the Korteweg-de Vries-Kawahara equation, which is a transport equation perturbed by dispersive terms of third and fifth order. It describes the evolution of small but finite amplitude long waves in various problems in fluid dynamics. Both the decaying case on the full line and the periodic case are considered. If the initial data $u|_{t=0} = u_0$ are of high regularity, $u_0 \in H^5(\mathbb{R})$, the schemes are shown to converge to a classical solution. Finally, the convergence is illustrated by an example.

Key words. Kawahara Equation, finite difference scheme, implicit schemes, convergence, existence.

1. Introduction

1.1. The Equation. This paper is concerned with the Korteweg-de Vries-Kawahara (Kawahara) equation, which reads

$$\begin{cases} u_t + uu_x + \partial_x^3 u = \partial_x^5 u, & (x,t) \in \Pi_T, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1}$$

where $\Pi_T = \mathbb{R} \times (0,T]$ with fixed $T > 0$, u_0 the given initial data, and $u : \Pi_T \mapsto \mathbb{R}$ is the unknown scalar map. It is well known that the one-dimensional waves of small but finite amplitude in dispersive systems (e.g., the magneto-acoustic waves in plasmas, the shallow water waves, the lattice waves, etc.) can be described by the Korteweg-de Vries (KdV) equation, given by

$$u_t + uu_x + \partial_x^3 u = 0, \tag{2}$$

which admits either compressive or rarefactive steady solitary wave solution (by a solitary water wave, we mean a travelling wave solution of the water wave equations for which the free surface approaches a constant height as $|x| \to \infty$) according to the sign of the dispersion term (the third order derivative term). In fact, in the galaxy of dispersive equations used to model waves phenomena, KdV equation is undoubtedly the brightest star.

However, under certain circumstances, it might happen that the coefficient of the third order derivative in the KdV equation becomes significantly small or even zero. In such a scenario, it is customary to take account of the higher order effect of dispersion in order to balance the nonlinear effect. As a result one may obtain a generalized nonlinear dispersive equation, known as Kawahara equation, which has a form of the KdV equation with an additional fifth order derivative term, given by (1). The Kawahara equation, an important nonlinear dispersive wave equation, describes solitary wave propagation in media in which the first order dispersion is anomalously small. A more specific physical background of this equation was introduced by Hunter and Scheurle [11], where they used it to describe the evolution of solitary waves in fluids in which the Bond number is less than but close to $\frac{1}{3}$ and

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the Froude number is close to 1. In the literature, this equation is also referred to as the fifth order KdV equation or singularly perturbed KdV equation. The fifth order term $\partial_x^5 u$ is called the Kawahara term.

1.2. Mathematical Background. There exists a fairly satisfactory well posedness theory for both KdV and Kawahara equations. The literature herein is substantial, and we will here only give a non-exhaustive overview. Within the existing framework, we mention the remarkable paper by Kenig et al., where the authors provide the local existence theory for the KdV equation in the Sobolev Space H^s, for $s > -3/4$. For a completely satisfactory well posedness theory for KdV equation, we refer to the monograph of Tao [23], and references therein.

Over the past four decades, there has been an increased interest to understand the solitary wave solutions of the Kawahara equation [6, 14, 16, 17]. It is found that, similar to the KdV equation, the Kawahara equation also has solitary wave solutions which decay rapidly to zero as $t \to \infty$, but unlike the KdV equation whose solitary wave solutions are non-oscillating, the solitary wave solutions of the Kawahara equation have oscillatory trails. This shows that the Kawahara equation is not only similar but also different from the KdV equation in the properties of solutions. The strong physical background of the Kawahara equation and such similarities and differences between it and the KdV equation in both the form and the behavior of the solution render the mathematical treatment of this equation particularly interesting. The Cauchy problem given by (1) has been studied by a few authors [3,7,15,24,25]. In that context, we mention the paper [3], where authors have shown that the problem (1) has a local solution $u \in C([-T,T]; H^r(\mathbb{R}))$ if $u_0 \in H^r(\mathbb{R})$ and $r > -1$. This local result combined with the energy conservation law yields that (1) has a global solution $u \in C([-\infty,\infty]; L^2(\mathbb{R}))$ if $u_0 \in L^2(\mathbb{R})$. Furthermore, the above mentioned results for (1) has been improved can be found in [25]. They even managed to prove local existence of solutions for $u_0 \in H^r(\mathbb{R})$, for $r \geq -7/5$ and global existence for $u_0 \in H^r(\mathbb{R})$, for $r > -1/2$. For the well posedness theory of (1), we refer to [7] and for the regularity results of such solutions, we refer to [20].

1.3. Numerical Approaches. There has been a number of papers involving the numerical computation of solutions of the Cauchy problem (1). For the KdV equation, a galore of numerical schemes available in literature. We just mention an interesting fact, and rarely referred to in the current literature, is that the first mathematical proof of existence and uniqueness of solutions of the KdV equation, was accomplished by Sjöberg [22] in 1970, using a finite difference approximation. His approach is based on a semi-discrete approximation where one discretizes the spatial variable, thereby reducing the equation to a system of ordinary differential equations. However, we stress that for numerical computations also this set of ordinary differential equations will have to be discretized in order to be solved. Therefore, to have a completely satisfactory numerical method, one seeks a fully discrete scheme that reduces the actual computation to a solution of a finite set of algebraic equations. In fact, this is accomplished in a recent paper by Holden et al. [8], both in the periodic case and on the full line.

A popular numerical approach has been the application of various spectral methods. Fourier-Galerkin spectral method for the KdV and Kawahara equations has been studied in [1, 18, 19]. Pseudospectral method or spectral collocation method have been used to solve PDEs like KdV, Kawahara equations in [4, 5]. On the other hand, in [18], an error estimate for a simple spectral fully discrete scheme