INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 13, Number 2, Pages 244–264

LINEAR AND QUADRATIC FINITE VOLUME METHODS ON TRIANGULAR MESHES FOR ELLIPTIC EQUATIONS WITH SINGULAR SOLUTIONS

GUANGHAO JIN, HENGGUANG LI, QINGHUI ZHANG, AND QINGSONG ZOU

Abstract. This paper is devoted to the presentation and analysis of some linear and quadratic finite volume (FV) schemes for elliptic problems with singular solutions due to the non-smoothness of the domain. Our FV schemes are constructed over specially-designed graded triangular meshes. We provide sharp parameter selection criteria for the graded mesh, such that both the linear and quadratic FV schemes achieve the optimal convergence rate approximating singular solutions in H^1. In addition, we show that on the same mesh, a linear FV scheme obtains the optimal rate of convergence in L^2. Numerical tests are provided to verify the analysis.

Key words. Finite volume method, singular solution, optimal convergence rate.

1. Introduction

With good local flux-conservation properties, the finite volume method (FVM) is used in a wide range of computations, especially in computational fluid dynamics (see [5, 25, 28, 30, 39, 40, 41, 44] and references therein). The mathematical theory of FVM [19, 30, 34] has not been fully developed, at least, not as satisfactory as that for the finite element method. Most works concentrate on linear or quadratic schemes on quasi-uniform meshes (see e.g., [4, 7, 18, 23, 34, 35, 45]). In addition, a few studies have been conducted for high order FV schemes. We here mention 1D high order FV schemes [8, 42], high order FV schemes over rectangular meshes [6, 46], and high order FV schemes over triangular meshes [11, 12]. These high order methods are efficient when the solution of the problem is sufficiently smooth.

It is well known that the solution of elliptic equations may have singularities due to the non-smoothness of the domain, even when the other given data are smooth. In particular, consider the Poisson problem

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{1}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$. Then, given $f \in H^{-1}(\Omega) = H_0^1(\Omega)'$, there exists a unique solution $u \in H_0^1(\Omega)$ to (1), defined by the variational form

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx, \qquad \forall\ v \in H_0^1(\Omega). \tag{2}$$

Received by the editors March 15, 2015.

2000 *Mathematics Subject Classification.* 65N08, 65N15, 65N50, 35J15.

H. Li was partially supported by the NSF Grants DMS-1158839, DMS-1418853, and the Wayne State University Grants Plus Program. Q. Zhang was partially supported by the Natural Science Foundation of China under grant 11001282 and 11471343, and Guangdong Provincial Natural Science Foundation of China under grant 2015A030306016. Q. Zou was partially supported by the National Natural Science Foundation of China through grants 11571384 and 11428103, and by Guangdong Provincial Natural Science Foundation of China under grant through grant 2014A030313179.

Correspondence to Qingsong Zou : mcszqs@mail.sysu.edu.cn.

In the case that the boundary $\partial\Omega$ is smooth, we have the *full regularity* estimate for the solution [17, 20, 36],

$$\|u\|_{H^{m+1}(\Omega)} \leq C\|f\|_{H^{m-1}(\Omega)}, \qquad m \geq 0, \tag{3}$$

where the constant $C > 0$ depends on the domain, but not on f. On a polygonal domain Ω, however, the full regularity result holds only in the interior region away from the vertices. On the entire domain Ω, the solution u may only belong to $H^{1+s}(\Omega)$ for a given smooth function f, where s is fixed and depends on the geometry of the boundary.

The singularity in the solution can significantly slow down the convergence rate of the numerical approximation, as well as raise concerns on the theoretical justification of the numerical scheme. Compared with the tremendous effort to develop optimal finite element algorithms [1, 2, 3, 22, 29, 33, 38, 43], fewer results are available on the FVMs for singular solution, and most of them only concern linear FV schemes. See [9, 15] and reference therein for some relevant works. In particular, three *linear* FVMs are proposed in [15] to approximate solutions of equation (1) with corner singularities. The mesh and dual mesh are carefully designed, such that the associated FV solutions achieve the optimal rate of convergence that is expected for smooth solutions.

In this paper, we develop new *linear* and *quadratic* FVMs approximating singular solutions of equation (1). In particular, we give a simple and explicit construction of graded meshes and the dual meshes, such that the associated linear and quadratic FV solutions achieve the optimal convergence rate in the H^1-norm. In addition, we will show that the L^2-convergence rate of the proposed linear FVM is also optimal. Our analysis is based on the stability of the FV schemes, sharp regularity estimates in suitable weighted Sobolev spaces, and rigorous interpolation error estimates in these spaces. These results extend to more general elliptic equations. It is also possible to apply the analytical tools developed here to other high order well-posed FVMs.

The rest of the paper is organized as follows. In Section 2, we introduce the linear and quadratic FV schemes and the graded triangular meshes. Determined by a set of grading parameters, these graded meshes have good geometric properties that will also be discussed. In Section 3, we present the detailed analysis in suitable function spaces and obtain the main result of the paper. In particular, we give regularity estimates, interpolation error estimates, and the continuity estimates of the FV bilinear forms. Using these results, in Theorem 3.9, we provide sharp parameter selection criteria for the graded mesh, such that the optimal convergence rate is recovered for the associated FV solutions in the H^1-norm. The L^2 error estimate for a linear FV algorithm is summarized in Corollary 3.11. In Section 4, we report numerical results from both linear and quadratic FV schemes. These results are in strong agreement with our theoretical prediction, and hence verify the theory.

Throughout the paper, by $a \simeq b$, we mean that there are constants $C_1, C_2 > 0$, independent of the mesh level, such that $C_1 b \leq a \leq C_2 b$. The generic constant $C > 0$ in our analysis below may be different at different occurrences. It will depend on the computational domain, but not on the functions involved in the estimates or the mesh level in the FV algorithms.