INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 13, Number 2, Pages 230–243

© 2016 Institute for Scientific
Computing and Information

A LITERATURE SURVEY OF MATHEMATICAL STUDY OF METAMATERIALS

JICHUN LI

Abstract. Since the successful construction of the so-called double negative metamaterials in 2000, there has been a growing interest in studying metamaterials across many disciplinaries. In this paper, we present a survey of recent progress in metamaterials and its applications from the mathematical point of view. Due to the great amount of papers published in this area, here we mainly discuss those issues interested to us. Our main goal is to attract more mathematicians to study this fascinating subject.

Key words. Metamaterials, Maxwell's equations, homogenization, edge elements, cloaking.

1. Introduction

In terms of metamaterials, we are specifically interested in those artificially structured composite materials with simultaneously negative electric permittivity ϵ and magnetic permeability μ.

According to Solymar and Shamonina [1, p.317], four seminar papers made the birth of the subject of metamaterials. The first one is by Russian physicist Victor Veselago [2], who wrote the fundamental paper on metamaterial (he called the left-hand material). In this paper, he investigated many properties unique to substances with both negative permittivity ϵ and negative permeability μ, even though nobody knew how to construct such a material at that time. The second important paper is due to the paper published in 2000 by David Smith *et al.* [3]. In this paper, through a physical experiment they demonstrated a composite medium (formed by a periodic array of interspaced conducting nonmagnetic split ring resonators and continuous wires) exhibits a frequency region in the microwave regime with simultaneously negative values of effective permeability $\mu_{eff}(\omega)$ and permittivity $\epsilon_{eff}(\omega)$, where ω is the frequency of incident radiation. This split ring structure forms the first successfully constructed left-handed medium, or double negative metamaterial. The third seminar paper is due to Shelby, Smith and Schultz [4], who in early 2001 presented experimental data at microwave frequencies on a structured metamaterial that there exists a frequency band where the effective index of refraction (normally defined as $n = \sqrt{\epsilon\mu}$) is negative when both ϵ and μ are negative. The other landmark work is due to John Pendry's perfect lens paper published in 2000 [5], in which he proposed the idea to use a slab of negative refractive index material to bring light to a perfect focus without the usual constraints imposed by wavelength. The principle behind this is that the negative refractive index material can restore not only the phase of propagating waves but also the amplitude of evanescent waves.

Since 2000, there has been a tremendous growing interest in studying metamaterial across many disciplines due to its potential revolutions in areas such as

Received by the editors March 10, 2015.
2000 *Mathematics Subject Classification.* 65M06, 92B05, 35Q92.
Work partially supported by NSFC Project 11271310, and NSF grant (DMS-1416742).

communications, sensing, radar technology, sub-wavelength imaging, data storage, and invisible cloak device.

2. Metamaterial Invisibility Cloaks

In June 23, 2006's issue of Science magazine, Leonhardt [6] and Pendry *et al.* [7] independently published their works on electromagnetic cloaking. Leonhardt used a conformal mapping to describe how an inhomogeneous indices of refraction $n(\boldsymbol{x})$ in two dimensions can cause light rays to go around a region and emerge on the other side as if they had passed through a free space where $n = 1$. While Pendry's idea is to enclose a finite size object to be cloaked by a specially designed metamaterial coating, which can control the electromagnetic field by mimicking the heterogeneous anisotropic nature of the matrices of permittivity and permeability obtained through changes of coordinates. In Nov.10, 2006's Science, Schurig *et al.* [8] demonstrated the first practical realization of such a cloak with the use of a metamaterial consisting of concentric layers of split ring resonators (SRRs) and made a copper cylinder invisible to an incident plane wave at a specific microwave frequency (8.5 GHz).

Actually, there are many other cloaking ideas with metamaterials. For example, Alu and Engheta [9] proposed an idea of employing a plasmonic or metamaterial cover to drastically reduce the overall scattering from moderately sized objects by means of a scattering cancellation effect. Later they [10] extended the idea to using a plasmonic coating to render an electromagnetic sensor almost invisible to detection by incident waves, while the sensor can remain effective as a device to receive, measure, and observe incident waves.

The idea of "cloaking by anomalous localized resonance" was proposed in 2006 by Milton and Nicorovici [11] and has been further developed by many researchers such as Bruno and Lintner[12], Bouchitte and Schweizer [13], Ammari *et al.* [14], Kohn *et al.* [15], and Nguyen [16]. As Kohn *et al.* [15] mentioned that the mathematical problem of this type cloaking boils down to the investigation of the behavior of the elliptic problem $\nabla \cdot (a(\boldsymbol{x})\nabla u(\boldsymbol{x})) = f(\boldsymbol{x})$, where $a(\boldsymbol{x})$ is a complex coefficient with a matrix-shell-core structure, with real part equal to 1 in the matrix and the core, and -1 in the shell. The interesting problem is to understand the resonant behavior of the solution when the imaginary part of $a(\boldsymbol{x})$ goes to zero, and how the location of the source f plays in the resonance. Many papers are restricted to radial geometries except [14, 15].

Among many proposed cloaking techniques with metamaterials, Pendry *et al.*'s cloaking technique [7] seems to be the most popular one, which is nicknamed as *transformation optics/electromagnetics* (e.g., [17, 18]). It is agreed now that the transformation based cloaking was first discovered back in 2003 by mathematicians Greenleaf *et al.* [19, 20, 21] for nondetectability examples in the context of the Calderón problem.

The principle behind transformation optics is to use a coordinate transformation to derive the spatial dependent permittivity and permeability to guide the wave. For electromagnetic wave, the derivation boils down to the important property that Maxwell's equations are form invariant under coordinate transformations.

Theorem 2.1. [22, Appendix A] *Consider the time-harmonic Maxwell's equations (assuming time harmonic variation of* $\exp(j\omega t)$*):*

$$\nabla \times \boldsymbol{E} + j\omega\mu\boldsymbol{H} = 0, \quad \nabla \times \boldsymbol{H} - j\omega\epsilon\boldsymbol{E} = 0, \tag{1}$$