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AN AUGMENTED IIM FOR HELMHOLTZ/POISSON EQUATIONS ON IRREGULAR DOMAINS IN COMPLEX SPACE

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Abstract. In this paper, an augmented immersed interface method has been developed for Helmholtz/Poisson equations on irregular domains in complex space. One of motivations of this paper is for simulations of wave scattering in different geometries. This paper is the first immersed interface method in complex space. The new method utilizes a combination of methodologies including the immersed interface method, a fast Fourier transform, augmented strategies, least squares interpolations, and the generalized minimal residual method (GMRES) for a Schur complement system, all in complex space. The new method is second order accurate in the L^∞ norm and requires $O(N\log(N))$ operations. Numerical examples are provided for a variety of real or complex wave numbers.

Key words. Helmholtz equation, complex space, irregular domain, augmented immerse interface method, fast Poisson solver in complex space.

1. Introduction

In this paper, we consider two-dimensional Helmholtz equations in complex space on irregular domains,

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+ku=f(x,y), \qquad (x,y)\in\Omega\subset R^2,\\
&u(x,y)\Big|_{\partial\Omega}=u_0(x,y),\\
&u(x,y):\Omega\to\mathbb{C};\quad f(x,y):\Omega\to\mathbb{C};\quad k\in\mathbb{C},
\end{aligned}
\tag{1}
$$

see Fig. 1 for an illustration, where the domain Ω is a half circle with two parts of the boundary $\partial\Omega_1$ and $\partial\Omega_2$. One particular application is numerical simulations of wave scattering when $k\geq 0$ on an uneven surface, see for example, [2, 10] and the references therein. In reality of the electromagnetic field models, the domains are discontinuous media with general interfaces and different complex wave numbers which represents both electric and magnetic charges. The irregularity in the domain presents extra challenges for researchers and engineers do develop fast and accurate numerical methods.

One difficulty with the problem is to deal with complex numbers. An intuitive approach is to separate the problem as the real and imaginary parts. If we define $u=v+iw$, $k=k_1+k_2i$, $f=f_1+if_2$, where $i=\sqrt{-1}$ is the imaginary unit, then

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the original partial differential equation (PDE) becomes a system of PDEs

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + k_1 v - k_2 w = f_1(x, y), \tag{2}$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + k_2 v + k_1 w = f_2(x, y). \tag{3}$$

It is not straightforward to such a system efficiently. A naive iterative method between the two system will lead to slow convergence or divergence. While it is possible to solve the coupled system of PDEs, it is more expensive than to solve it directly using the original form in complex number space.

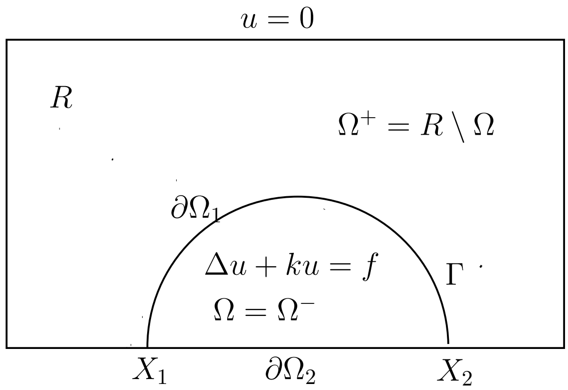

FIGURE 1. A diagram of the set-up of the problem. The Helmholtz equation is defined inside the half circle that is embedded in a rectangle R. We denote the domain inside R and outside of Ω as $\Omega^+ = R \setminus \Omega$. We also denote the original domain Ω as Ω^- for convenience. The part of boundary $\partial\Omega_1$ excluding the two points $\mathbf{X}_1$ and $\mathbf{X}_2$ becomes an interface that is denoted as Γ.

Our strategy is to solve the problem directly using complex numbers based on the augmented immersed interface method (AIIM) [4, 7]. The immersed interface method [5] is based on Cartesian grid method similar to the immersed boundary (IB) method [11, 12]. While there may be different methods in the literature, in this paper, we present a fast second order accurate finite difference method based on the immersed interface method (IIM) in complex number space.

In our approach, we use an imbedding technique to put the irregular domain Ω into a rectangular domain R so that a uniform Cartesian mesh can be used, and a fast Helmholtz/Poisson solver such as one that based on a discrete fast Fourier transform (FFT) can be utilized. We also extend the source term $f(x, y)$ to the entire rectangular domain by zero. Then the irregular boundary will become in interface. We set zero boundary condition at the auxiliary rectangular domain. Thus the original problem can be treated as an interface problem if we know the jump conditions in the solution and the flux. We set the jump in the solution as an augmented variable $[u] = Q$ and let $[u_n] = [\frac{\partial u}{\partial n}] = 0$. The augmented variable should be chosen such that the Dirichlet boundary condition is satisfied along $\partial\Omega$. The augmented variable has co-dimension one compared with that of the solution.