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MONOTONICITY/SYMMETRICITY PRESERVING RATIONAL QUADRATIC FRACTAL INTERPOLATION SURFACES

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Abstract. This paper presents the theory of $\mathcal{C}^1$-rational quadratic fractal interpolation surfaces (FISs) over a rectangular grid. First we approximate the original function along the grid lines of interpolation domain by using the univariate $\mathcal{C}^1$-rational quadratic fractal interpolation functions (fractal boundary curves). Then we construct the rational quadratic FIS as a blending combination with the x-direction and y-direction fractal boundary curves. The developed rational quadratic FISs are monotonic whenever the corresponding fractal boundary curves are monotonic. We derive the optimal range for the scaling parameters in both positive and negative directions such that the rational quadratic fractal boundary curves are monotonic in nature. The relation between x-direction and y-direction scaling matrices is deduced for symmetric rational quadratic FISs for symmetric surface data. The presence of scaling parameters in the fractal boundary curves helps us to get a wide variety of monotonic/symmetric rational quadratic FISs without altering the given surface data. Numerical examples are provided to demonstrate the comprehensive performance of the rational quadratic FIS in fitting a monotonic/symmetric surface data. The convergence analysis of the monotonic rational quadratic FIS to the original function is reported.

Key words. Fractals, Fractal Interpolation Functions, Rational Quadratic Fractal Interpolation Surfaces, Convergence, Monotonicity, Symmetricity.

1. Introduction

The field of fractals [21] is introduced as an interdisciplinary area between branches of mathematics and physics, and later applied successfully in different areas of science and engineering. Fractals provide a powerful and effective tool to approximate projections of physical objects such as coastlines, profiles of mountains, plants as well as experimental data that have non-integer dimension. To provide an alternative tool for traditional interpolants, Barnsley [3] introduced the concept of fractal interpolation functions (FIFs) via iterated function system (IFS). A FIF contains a set of free variables called the scaling parameters. The variation of scaling parameters helps us to generate a wide variety of smooth or non-smooth FIFs for the same interpolation data. Restricting the scaling parameters with respect to the horizontal scaling parameters, Barnsley and Harrington [4] developed a method to construct a differentiable FIF that interpolates the prescribed data if the values of derivatives of an original function are known at the initial end-point of the interval. The fractal polynomial splines with general type of boundary conditions are studied recently by (i) constructive approach in [9, 14] (ii) α-fractal functions in [11, 22].

The study of fractal surfaces are useful in scientific applications such as image processing [23], geology [15], chemistry [24], etc. Geronimo and Hardin [18] developed the fractal interpolation surface on flexible domains. Simultaneously, by using barycentric co-ordinates, Zhao [26] gave two algorithms that generalize the earlier construction described in [18]. Xie and Sun [25] constructed a bivariate FIS

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on rectangular grids with arbitrary contraction factors and without any restriction on boundary data. Dalla [16] extemporized this construction by using collinear boundary points, and demonstrated that the attractor is a continuous FIS. Subsequent developments in this direction are carried out by Bouboulis and Dalla [6], Chand and Navascués [10], Feng et al. [17], Chand [12]. However, the constructions mentioned above may not produce the monotonic/symmetric fractal surfaces even if the given surface data is monotonic/symmetric.

Although the field of interpolation has been cultivated for centuries, the demand for more effective tools is very much active due to the modeling problems in complexity, and the manufacturing requirements in the early stage of surface design. Spline representation to visualize a scientific data is of great significance in computer graphics, geometric modeling, and numerical analysis. Although splines are smooth, they may not fulfill the user's qualitative requirements. For instance, the given data may be generated from a monotone/symmetric surface but the resulting interpolant may not satisfy these properties, and induce artificial or exaggerated hills and valleys in the interpolating surface. For the case of surface generation, several non-fractal methods have been proposed by a number of authors which preserves properties such as positivity, monotonicity and/or convexity of the data. Beatson and Ziegler [5] presented a monotonicity preserving surface interpolant over a triangular grid. This surface is uniquely determined by the functional values and first order partial derivatives at the vertices of the triangular grid. Asaturyan and Unswoth [1] developed a monotonicity preserving biquadratic splines over rectangular grids. In their approach, a modification at x-location of one edge of a sub-rectangle gives a variation throughout the grid for all sub-rectangle edges located at the original x-values, and hence the scheme is global. By developing the necessary and sufficient conditions on the first and mixed partial derivatives at grids, Carlson and Fritsch [7] produced a monotonic surface interpolant over rectangular grid. Kouibia and Pasadas [20] presented an approximation problem of parametric curves and surfaces from the Lagrange or Hermite data set. However, the shape preserving interpolation technique for the surface generation problem via fractal technique is not yet initiated. This paper specifically concentrates on the visualization of the monotonic/symmetric surface data arranged on a rectangular grid in the form of rational quadratic FISs.

In order to show the deficiency of the classical blending $\mathcal{C}^1$- cubic spline surface scheme, consider a monotonically increasing and symmetric surface data as in Table 1. For simplicity of presentation, we have used triplet $(.,.,.)$, where the first component indicates function value and second, third components represent the first order partial derivatives with respect to x-direction and y-direction, all are evaluated at the typical point (x_i, y_j). It can be easily seen that although surface data in Table 1 is increasing, the classical surface in Fig. 1 is not increasing.

Table 1. Monotonically increasing symmetric surface data.

$\downarrow x/y \rightarrow$	1	2	3	4
1	(1,1,1)	(2,4,2)	(3,9,3)	(4,16,4)
2	(2, 2,4)	(4, 8, 8)	(6, 18, 12)	(8,32,16)
3	(3, 3, 9)	(6, 12, 18)	(9, 27, 27)	(12, 48, 36)
4	(4, 4, 16)	(8, 16, 32)	(12, 36, 48)	(16, 64, 64)

Not only this blending surface scheme, but also several classical and fractal surface interpolation schemes do not preserve the monotonicity attached with given