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LOCAL AND PARALLEL FINITE ELEMENT ALGORITHM BASED ON MULTILEVEL DISCRETIZATION FOR EIGENVALUE PROBLEMS

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Abstract. In this paper, a local and parallel algorithm based on the multilevel discretization is proposed for solving the eigenvalue problem by the finite element method. With this new scheme, the eigenvalue problem solving in the finest grid is transferred to solutions of the eigenvalue problems on the coarsest mesh and a series of solutions of boundary value problems on each level mesh. Therefore this type of multilevel local and parallel method improves the overall efficiency of solving the eigenvalue problem. Some numerical experiments are presented to validate the efficiency of the new method.

Key words. eigenvalue problem, multigrid, multilevel correction, local and parallel method, finite element method.

1. Introduction

Solving large scale eigenvalue problems becomes a fundamental problem in modern science and engineering society. However, it is always a very difficult task to solve high-dimensional eigenvalue problems which arise from physical and chemistry sciences. Xu and Zhou [28] give a type of two-grid discretization method to improve the efficiency of the solution of eigenvalue problems. By the two-grid method, the solution of eigenvalue problem on a fine mesh is reduced to a solution of eigenvalue problem on a coarse mesh (which depends on the fine mesh) and a solution of the corresponding boundary value problem on the fine mesh [28]. For more details, please read [25, 26]. Combing the two-grid idea and the local and parallel finite element technique [27], a type of local and parallel finite element technique to solve the eigenvalue problems is given in [29] (also see [10]). Recently, a new type of multilevel correction method for solving eigenvalue problems which can be implemented on multilevel grids is proposed in [14]. In the multilevel correction scheme, the solution of eigenvalue problem on the finest mesh can be reduced to a series of solutions of the eigenvalue problem on a very coarse mesh (independent of the finest mesh) and a series of solutions of the boundary value problems on the multilevel meshes. The multilevel correction method gives a way to construct a type of multigrid scheme for the eigenvalue problem [15].

In this paper, we propose a type of multilevel local and parallel scheme to solve the eigenvalue problem based on the combination of the multilevel correction method and the local and parallel technique. The special property of this scheme

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is that we can do the local and parallel computing for any level grids and the mesh size of the original coarse triangulation is independent of the finest triangulation. With this new method, the solution of the eigenvalue problem is not going to be more difficult than the solution of the boundary value problems by the local and parallel algorithm since the main part of the computation in the multilevel local and parallel method is solving the boundary value problems.

The standard Galerkin finite element method for eigenvalue problems has been extensively investigated, e.g. Babuška and Osborn [2, 3], Chatelin [8] and references cited therein. There also exists analysis for the local and parallel finite element method for the boundary value problems and eigenvalue problems [10, 21, 22, 27, 28, 29]. Here we adopt some basic results in these papers for our analysis. The corresponding error and computational work estimates of the proposed multilevel local and parallel scheme for the eigenvalue problem will be analyzed. Based on the analysis, the new method can obtain optimal errors with an optimal computational work in each processor.

An outline of this paper goes as follows. In the next section, a basic theory about the local error estimate of the finite element method is introduced. In Section 3, we introduce the finite element method for the eigenvalue problem and the corresponding error estimates. A local and parallel type of one correction step and multilevel correction algorithm will be given in Section 4. The estimate of the computational work for the multilevel local and parallel algorithm is presented in section 5. In Section 6, three numerical examples are presented to validate our theoretical analysis and some concluding remarks are given in the last section.

2. Discretization by the finite element method

In this section, we introduce some notation and error estimates of the finite element approximation of linear elliptic problems. The letter C (with or without subscripts) denotes a generic positive constant which may be different at its different occurrences throughout the paper. For convenience, the symbols $\lesssim$, $\gtrsim$ and $\approx$ will be used in this paper. That $x_1 \lesssim y_1, x_2 \gtrsim y_2$ and $x_3 \approx y_3$, mean that $x_1 \le C_1 y_1$, $x_2 \ge c_2 y_2$ and $c_3 x_3 \le y_3 \le C_3 x_3$ for some constants C_1, c_2, c_3 and C_3 that are independent of mesh sizes (see, e.g., [24]). We shall use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ and their associated norms and seminorms (see, e.g., [1]). For $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega)$ and $H_0^1(\Omega) = \{v \in H^1(\Omega) : \ v|_{\partial\Omega} = 0\}$, where $v|_{\partial\Omega} = 0$ is in the sense of trace, $\|\cdot\|_{s,\Omega} = \|\cdot\|_{s,2,\Omega}$.

For $G \subset D \subset \Omega$, the notation $G \subset\subset D$ means that $\operatorname{dist}(\partial D \setminus \partial\Omega, \partial G \setminus \partial\Omega) > 0$ (see Figure 1). It is well known that any $w \in H_0^1(\Omega_0)$ can be naturally extended to be a function in $H_0^1(\Omega)$ with zero outside of Ω_0, where $\Omega_0 \subset \Omega$. Thus we will show this fact by the abused notation $H_0^1(\Omega_0) \subset H_0^1(\Omega)$.

2.1. Finite element space. Now, let us define the finite element space. First we generate a shape-regular decomposition $\mathcal{T}_h(\Omega)$ of the computing domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ into triangles or rectangles for $d = 2$ (tetrahedrons or hexahedrons for $d = 3$). The diameter of a cell $K \in \mathcal{T}_h(\Omega)$ is denoted by h_K. The mesh size function is denoted by $h(x)$ whose value is the diameter h_K of the element K including x.

For generality, following [27, 29], we shall consider a class of finite element spaces that satisfy certain assumptions. Now we describe such assumptions.

A.0. There exists $\gamma > 1$ such that

$$h_\Omega^\gamma \lesssim h(x), \qquad \forall x \in \Omega,$$

where $h_\Omega = \max_{x\in\Omega} h(x)$ is the largest mesh size of $\mathcal{T}_h(\Omega)$.