INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 13, Number 1, Pages 58–72

CONVERGENCE ANALYSES OF CRANK-NICOLSON ORTHOGONAL SPLINE COLLOCATION METHODS FOR LINEAR PARABOLIC PROBLEMS IN TWO SPACE VARIABLES

MORRAKOT KHEBCHAREON, AMIYA KUMAR PANI, AND GRAEME FAIRWEATHER

Abstract. The Crank-Nicolson (CN) orthogonal spline collocation method and its alternating direction implicit (ADI) counterpart are considered for the approximate solution of a class of linear parabolic problems in two space variables. It is proved that both methods are second order accurate in time and of optimal order in certain H^j norms in space. Also, L^∞ estimates in space are derived.

Key words. parabolic problems, orthogonal spline collocation, Crank-Nicolson method, alternating direction implicit method, optimal global error estimates.

1. Introduction

Consider the initial/boundary value problem comprising

$$\frac{\partial u}{\partial t} + Lu = f(x,y,t), \quad (x,y,t) \in \Omega_T \equiv \Omega \times (0,T], \tag{1}$$

the initial condition,

$$u(x,y,0) = u_0(x,y), \quad (x,y) \in \Omega, \tag{2}$$

and the Dirichlet boundary condition,

$$u(x,y,t) = 0, \quad (x,y,t) \in \partial\Omega \times (0,T], \tag{3}$$

where $\Omega = (0,1) \times (0,1)$ with boundary $\partial\Omega$. Here, $u_0(x,y)$ and $f(x,y,t)$ are given functions in their respective domains of definition, and $L = L_1 + L_2$ with

$$L_1 u = -a_1(x,y,t)u_{xx} + b_1(x,y,t)u_x + c(x,y,t)u, \tag{4}$$

$$L_2 u = -a_2(x,y,t)u_{yy} + b_2(x,y,t)u_y, \tag{5}$$

where

$$0 \le a_{\min} \le a_1(x,y,t), \ \ a_2(x,y,t) \le a_{\max}, \quad (x,y,t) \in \Omega_T.$$

For the approximate solution of this problem, we examine the Crank–Nicolson orthogonal spline collocation (OSC) scheme. In this method, OSC with C^1 piecewise polynomials of arbitrary degree $r \ge 3$ in each space variable is used for the spatial discretization and the resulting system of ordinary differential equations in the time variable is discretized using the trapezoid rule. We also consider an alternating direction implicit (ADI) version of this method. These methods are not new but, to the best of the authors' knowledge, a comprehensive convergence analysis of them has not yet appeared in the literature. Numerical experiments reported in the literature exhibit the expected second order accuracy in time and optimal order error estimates in various norms in space at each time step. In [9], the first convergence analysis of the Crank-Nicolson OSC method was presented for semilinear problems of the form (1)–(3); that is, problems in which the function f depends on the solution u. An algebraically linear form of this method, commonly known as

Received by the editors September 13, 2015.
2000 *Mathematics Subject Classification.* 65M15,65M70.

the extrapolated Crank-Nicolson method, was also considered. Both methods were proved to be second order accurate in time and of optimal accuracy in the L^2 norm in space. In the 1980s, ADI OSC methods were developed by various authors and used to solve practical problems modeled by parabolic equations in several space variables; see [2] for an overview of these methods. The first analysis of an ADI OSC method was given by Fernandes and Fairweather [7], who considered the ADI Crank-Nicolson OSC method for the heat equation in two space variables. Optimal L^2 and H^1 error estimates in space and second order accuracy in time were derived. An optimal H^2 error estimate for this method is a consequence of analysis given in [11].

In [3], Bialecki and Fernandes considered an ADI Crank-Nicolson OSC for (1)–(3) for the case in which $b_1 = b_2 = c = 0$ and $r = 3$. They proved that the scheme is second order accurate in time and third order accurate in space in a norm which is stronger than the L^2 norm but weaker than the H^1 norm. In [4], the ADI Crank-Nicolson OSC was considered for (1)–(3) with $r = 3$ and proved to be second order in time and of optimal accuracy in space in the H^1 norm. The authors state that the analysis can be easily extended to the case $r > 3$. For an overview of these methods and ADI OSC methods for other equations, see [8].

The primary purpose of this paper is to provide a complete convergence analysis of the Crank-Nicolson OSC method and the ADI Crank-Nicolson OSC method for the approximate solution of (1)–(3). Specifically, we prove that each method is second order accurate in time and of optimal accuracy in H^j norms in space. Moreover, some L^∞ estimates in space are obtained. An outline of the paper is as follows. In section 2, we introduce standard notation and basic lemmas used in the formulation and analysis of OSC methods. In section 3, the Crank-Nicolson OSC scheme is described and the optimal error estimates are derived. The ADI Crank-Nicolson OSC scheme is considered in section 4, and concluding remarks are presented in section 5.

2. Preliminaries and Basic Results

Set $I = (0,1)$, and let $\delta_x = \{x_i\}_{i=0}^{N_x}$ and $\delta_y = \{y_j\}_{j=0}^{N_y}$ be two partitions of $\bar{I}$ such that

$$0 = x_0 < x_1 < \cdots < x_{N_x-1} < x_{N_x} = 1, \qquad 0 = y_0 < y_1 < \cdots < y_{N_y-1} < y_{N_y} = 1.$$

Assume that the partition $\delta = \delta_x \otimes \delta_y$ of Ω is quasi-uniform. For $1 \le i \le N_x,\ 1 \le j \le N_y$, let

$$\begin{aligned} I_i^x &= (x_{i-1}, x_i), \quad I_j^y = (y_{j-1}, y_j), \quad I_{ij} = I_i^x \times I_j^y, \\ h_i^x &= x_i - x_{i-1}, \quad h_j^y = y_j - y_{j-1}, \end{aligned}$$

and set

$$h = \max(\max_i h_i^x, \max_j h_j^y).$$

Let $\mathcal{M}_r(\delta_x)$ and $\mathcal{M}_r^0(\delta_x)$ be the spaces of piecewise polynomials of degree $\le r$ with $r \ge 3$ defined by

$$\begin{aligned} \mathcal{M}_r(\delta_x) &= \{v | v \in C^1[0,1], v|_{I_i^x} \in P_r,\ 1 \le i \le N_x\}, \\ \mathcal{M}_r^0(\delta_x) &= \{v | v \in \mathcal{M}_r(\delta_x),\ v(0) = v(1) = 0\}, \end{aligned}$$

where P_r denotes the set of polynomials of degree $\le r$. The spaces $\mathcal{M}_r(\delta_y)$ and $\mathcal{M}_r^0(\delta_y)$ are defined similarly. Set

$$\mathcal{M}_r^0(\delta) = \mathcal{M}_r^0(\delta_x) \otimes \mathcal{M}_r^0(\delta_y).$$