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ON THE SINGULARLY PERTURBED SEMILINEAR REACTION-DIFFUSION PROBLEM AND ITS NUMERICAL SOLUTION

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Abstract. We obtain improved derivative estimates for the solution of the semilinear singularly perturbed reaction-diffusion problem in one dimension. This enables us to modify the transition points between the fine and coarse parts of the Shishkin discretization mesh. We prove that the numerical solution, obtained by using the central finite-difference scheme on the modified mesh, retains the same order of convergence uniform in the perturbation parameter as on the standard Shishkin mesh. However, the modified mesh may be denser in the layers than the standard one, and, when this is the case, numerical results show an improvement in the accuracy of the computed solution.

Key words. singularly perturbed boundary-value problem, reaction-diffusion, Shishkin mesh, finite differences, and uniform convergence.

1. Introduction

We consider the semilinear singularly perturbed boundary-value problem

$$(1)\qquad \begin{gathered} Tu := -\varepsilon^2 u'' + b(x,u) = 0, \quad x \in I := [0,1], \quad u(0) = u(1) = 0, \\ b_u(x,u) \geq \beta^2 > 0, \quad x \in I, \quad u \in D, \quad \beta > 0, \end{gathered}$$

where $0 < \varepsilon \leq \varepsilon^* \ll 1$, b is a sufficiently smooth function, $b \in C^k(I \times D)$, $k \geq 0$, and D is some closed bounded domain which we specify in Section 2. The problem has a unique solution $u \in C^{k+2}(I)$ for which the following derivative estimates hold true (cf. [17]):

$$(2)\qquad |u^{(i)}(x)| \leq M \left(1 + \varepsilon^{-i} e^{-\beta x/\varepsilon} + \varepsilon^{-i} e^{\beta(x-1)/\varepsilon}\right), \quad i = 0, 1, \ldots, k, \quad x \in I,$$

with M denoting a generic positive constant independent of ε. The estimates show that, in general, the solution u has boundary layers near $x = 0$ and $x = 1$.

Numerical methods for problems of type (1), sometimes in the linear version, are studied in [1, 2, 3, 8, 9, 11, 13, 14, 16, 17, 19]. The semilinear problem $Tu = 0$ is considered in [6, 15, 7] under relaxed conditions on b, which allow for multiple solutions with boundary or interior layers. We do not consider these relaxed conditions here. Instead, we focus on the condition on b stated in (1), which is assumed in most of the above-cited works. Our aim is to show that even with this condition it is possible to improve numerical results obtained when the problem (1) is discretized on a mesh of Shishkin type.

Shishkin meshes [3, 11, 9, 13] are arguably the most popular meshes for discretizing singular perturbation problems. The presence of layers is characteristic of solutions to singularly perturbed boundary-value problems and Shishkin meshes are layer-adapted. This is why they enable ε-uniform convergence of numerical

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solutions, which is the main goal of numerical methods for singularly perturbed problems. For the problem (1), the Shishkin mesh is divided into two fine parts in the layers and the coarse one outside the layers. The points at which the mesh step size changes are called transition points. The standard definition of the left transition point for (1) is $a\varepsilon \ln N/\beta$ and the right transition point is $1 - a\varepsilon \ln N/\beta$. The quantities N and a in this definition are respectively the total number of mesh steps and a sufficiently large positive parameter, which is related to the order of convergence of the numerical method. In general, the influence of the choice of the transition points and the complete mechanism of the Shishkin mesh are explained in details in [5]. A discussion of generalizations of the Shishkin mesh can be found in [9].

The crucial result of this paper is the modification of the estimates (2) to

$$|u^{(i)}(x)| \le M\left(1 + \varepsilon^{-i}e^{-\beta_0 x/\varepsilon} + \varepsilon^{-i}e^{\beta_1(x-1)/\varepsilon}\right), \quad i = 0, 1, \ldots, k, \quad x \in I, \tag{3}$$

where $\beta_i > 0$ and $b_u(i, u) > \beta_i^2$, $i = 0, 1$, for $u \in D$. This is obtained without any additional conditions on (1). The estimates in (3) may be sharper than those in (2). They also enable a redefinition of the transition points as $a\varepsilon \ln N/\beta_0$ and $1 - a\varepsilon \ln N/\beta_1$. It immediately follows that the standard central discretization of (1) on this modified Shishkin mesh (with $a \ge 2$) yields ε-uniform pointwise convergence of order almost 2. This is the same result as on the standard Shishkin mesh. However, it is possible that $\beta_i > \beta$, $i = 0, 1$, and we may get a better layer-resolving mesh since the transition points are moved closer to the end points where the layers occur. If this happens, the density of mesh points in the layers increases, because of which we can expect more accurate numerical results. This expectation has already been confirmed in [19] for the linear case of the problem (1).

The motivation for [19] and the present paper comes from [10], where a similarly modified transition point is used in numerical experiments with the quasilinear singularly perturbed convection-diffusion problem. However, the theoretical analysis in [10] is carried out for the standard Shishkin mesh, since no improved derivative estimates of the solution were available for use.

The outline of the paper is as follows. In Section 2 we analyze the continuous solution of the problem (1). We prove the estimates in (3), as well as some other estimates. Then, in Section 3 we consider the linear case of the problem and improve the results from Section 2. Section 3 also contains a discussion of the proof technique used for the semilinear problem (1) and the one in [19] for the linear problem. This shows that our present analysis is not a straightforward generalization of the analysis in [19]. In Section 4, the modified Shishkin mesh is defined and the ε-uniform convergence result for the central discretization scheme is proved. This is followed by Section 5, where we show that the piecewise-linear interpolation of the numerical solution retains the accuracy of the numerical solution. The results of sections 4 and 5 for the linear case are sharper than the results in [19]. Finally, Section 6 provides results of numerical experiments, which demonstrate improvements in the computed solution when compared to the results on the standard Shishkin mesh.

2. The general continuous problem

We assume that there exist constants u_* and u^* such that

$$u_* \le 0 \le u^*, \quad b(x, u_*) \le 0 \le b(x, u^*), \quad x \in I,$$