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ANALYSIS OF A SECOND-ORDER, UNCONDITIONALLY STABLE, PARTITIONED METHOD FOR THE EVOLUTIONARY STOKES-DARCY MODEL

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Abstract. We propose and analyze a partitioned numerical method for the fully evolutionary Stokes-Darcy equations that model the coupling between surface and groundwater flows. The proposed method uncouples the surface from the groundwater flow by using the implicit-explicit combination of the Crank-Nicolson and Leapfrog methods for the discretization in time with added stabilization terms. We prove that the method is asymptotically, unconditionally stable—requiring no time step condition—and second-order accurate in time with optimal rates in space. We verify the method's unconditional stability and second-order accuracy numerically.

Key words. Stokes, Darcy, groundwater, surface water, partitioned, decoupled, second-order accuracy, unconditional stability, asymptotic stability

1. Introduction

One of the difficulties in solving the Stokes-Darcy problem arises from the coupling of two different physical processes in two adjacent domains. Using partitioned methods to uncouple the Stokes from the Darcy equations resolves this issue and allows one to leverage existing algorithms already optimized to solve the physical processes in each subdomain. The first partitioned methods (first-order accurate) for the evolutionary Stokes-Darcy equations were studied in [19]. Other first-order partitioned methods were analyzed in [17], and second-order, long-time accurate, partitioned methods in [6]. In [15], it was shown that the implicit-explicit combination of the Crank-Nicolson and Leapfrog methods (CNLF) results in a second-order partitioned method for the Stokes-Darcy system. However, the conditional stability of CNLF makes the method impractical when faced with certain small-value model parameters.

By adding appropriate stabilization terms to both the Stokes as well as the groundwater flow equation, the proposed numerical scheme, denoted CNLF-stab and introduced in Section 3, equations (19)-(21), is unconditionally, asymptotically stable, as well as second-order convergent. More specifically, we prove that the added stabilization terms eliminate the time step restriction without affecting the second-order accuracy of the method. Further, we show that CNLF-stab controls the unstable mode due to Leapfrog and is thus asymptotically stable.

We let Ω_f, Ω_p denote two regular, bounded domains, the fluid and porous media regions respectively, and assume they lie across an interface I (Figure 1). Suppose that an incompressible fluid flows both ways across I, described by time-dependent Stokes flow in Ω_f and the groundwater flow equation in Ω_p. The fluid velocity field $u = u(x,t)$ and pressure $p = p(x,t)$, defined in Ω_f, and porous media hydraulic

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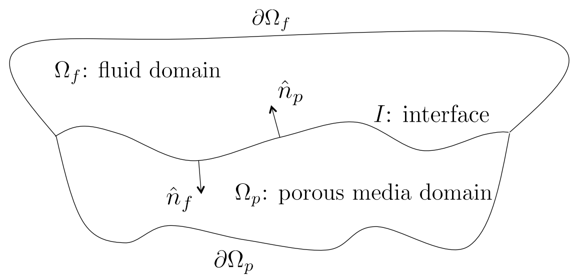

FIGURE 1. Fluid and porous media domains

head $\phi = \phi(x,t)$, defined in Ω_p, satisfy

$$\begin{aligned}
u_t - \nu\Delta u + \nabla p = f_f(x,t), \nabla\cdot u = 0,\ \text{in } \Omega_f,\\
S_0\phi_t - \nabla\cdot(\mathcal{K}\nabla\phi) = f_p(x,t),\ \text{in } \Omega_p,\\
u(x,0) = u_0,\ \text{in } \Omega_f,\ \phi(x,0) = \phi_0,\ \text{in } \Omega_p,\\
u(x,t) = 0,\ \text{in } \partial\Omega_f\backslash I, \phi(x,t) = 0,\ \text{in } \partial\Omega_p\backslash I,\\
+ \text{ coupling conditions across } I,
\end{aligned} \tag{1}$$

where the pressure, p, and the body forces in the fluid region, f_f, have been normalized by the fluid density, ρ. Denoted by f_p are the sinks or sources in the porous media region, $\nu > 0$ is the kinematic viscosity of the fluid, and $\mathcal{K}$ is the hydraulic conductivity tensor, assumed to be symmetric, positive definite with spectrum$(\mathcal{K}) \in [k_{\min}, k_{\max}]$. We assume Dirichlet boundary conditions at the exterior boundaries of the two domains (not including the interface I). We discuss the assumed coupling conditions in Section 2.

In the aforementioned equations, S_0 is the specific storage, defined as the volume of water that a portion of a fully saturated porous medium releases from storage per unit volume and per unit drop in hydraulic head, see [9, 11]. Table 1 gives values of S_0 for different materials [8, 13]. The time step condition for stability in regular CNLF, derived in [15], is

$$\Delta t \leq C\max\{\min\{h^2, gS_0\}, \min\{h, gS_0h\}\},$$

where g is the gravitational acceleration constant, h the mesh size in the finite element discretization, and C a positive constant independent of both h and Δt. The time step condition is sensitive to values of S_0 and this can be computationally restrictive in certain cases. For instance, since $g = O(10^1)$, if $S_0 \leq \mathcal{O}(10^{-3})$ and $h = \mathcal{O}(10^{-1})$, then the time step condition implies that $\Delta t \leq \mathcal{O}(S_0)$. A small time step is prohibitive since studying flow in large aquifers with low conductivity necessitates accurate calculations over long-time periods.

Another important parameter in our analysis is the hydraulic conductivity tensor, $\mathcal{K}$. In exact arithmetic, stability of CNLF does not depend upon $\mathcal{K}$. Since the hydraulic conductivity is often very small (Table 2, [1]), and computations are required over long-time intervals, unconditional stability—independent of $\mathcal{K}$—of our numerical scheme is desirable.

In Section 2 we present necessary preliminaries and the equivalent weak formulation of the Stokes-Darcy problem. In Section 3 we introduce the CNLF-stab method for the evolutionary Stokes-Darcy model and present the proof for unconditional, asymptotic stability. We prove second-order convergence of the method in