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A STABILIZED CHARACTERISTIC FINITE ELEMENT METHOD FOR THE VISCOELASTIC OLDROYD FLUID MOTION PROBLEM

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Abstract. In this article, a characteristic scheme is considered for the viscoelastic Oldroyd fluid flows based on the lowest equal-order finite element pair. The diffusion term in these equations is discretized by using finite element method, the temporal differentiation and advection terms are treated by characteristic scheme and the integral term is handled by applying right rectangle rule. Unconditionally stability and optimal error estimates for the velocity and pressure are derived. Finally, some numerical results are provided to verify the performance of the proposed method.

Key words. viscoelastic Oldroyd fluid motion problem, characteristic scheme, stabilized method, stability, error estimate.

1. Introduction

In this work, let Ω be an open bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Consider the following viscoelastic Oldroyd fluid flows

$$u_t - \nu\Delta u + \nabla p + (u\cdot\nabla)u - \int_0^t \rho e^{-\delta(t-s)}\Delta u ds = f, \tag{1}$$

with $x \in \Omega,\ t \in (0,T]$ and incompressible condition

$$\text{div } u(t,x) = 0 \qquad \forall\ t \in (0,T], \quad x \in \Omega, \tag{2}$$

and the initial and boundary conditions

$$u(x,0) = u_0(x) \quad x \in \Omega; \quad u|_{\partial\Omega} = 0 \quad \text{for all } t \in (0,T], \tag{3}$$

where $\rho \geq 0,\ 1/\delta$ is the relaxation time, u represents the velocity, p the pressure, f the prescribed external force, $u_0(x)$ the initial velocity, ν is the viscosity and $T > 0$ is a finite time.

From the expressions of equations (1)-(3), we know that (1)-(3) are the generalization of the initial boundary value problem for the Navier-Stokes equations, and equations (1)-(3) are used as model in viscoelastic Oldroyd flows [15, 18]. The importance of ensuring the compatibility of the approximations of velocity and pressure by satisfying the *inf-sup* condition is widely known in [7]. Although stable mixed finite element pairs have been studied over the years [13, 16, 19], the low order finite element pairs not satisfying the *inf-sup* condition may work well not only in theoretical filed but also in computation (see [2, 3, 8, 14, 24] and the reference therein). In these stabilized techniques, polynomial pressure projection method which developed in [2] is the most attractive due to the following reasons: (i) The method does not require the approximation of the pressure derivatives and

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the mesh-dependent parameters. (ii) The method is unconditionally stable. (iii) The method can be applied to the existing codes with a little additional effort. Therefore, much attention has been attracted to solve various kinds of problems by using this stabilized technique, for example, we can refer to [14, 20, 22, 26].

On the other hand, characteristic scheme is designed to deal with convection-diffusion problems. This scheme can treat the convection-dominated equations efficiently [1, 3, 6]. Characteristic scheme has also been applied to solve the incompressible flow problems. For example, Pironneau analyzed the Navier-Stokes equations and obtain the suboptimal convergence results in [17], Suli in [19] improved the results of [17], Zhang et al. considered incompressible flows by combining stabilized method with characteristic scheme in [23, 25] and [4, 6, 21] for convection-dominated problems.

In this work, we try to combine the modified method of characteristics with stabilized method to treat the viscoelastic Oldroyd flows. The combination is efficient and keeps the advantages of two methods and avoids their deficits. The main contribution of this article is to establish the stability and convergence of the stabilized characteristic finite element solutions based on the uniqueness condition.

The rest of this paper is organized as follows. In Section 2, the notations and some basic results for equations (1)-(3) are recalled. In Section 3, we provide the boundedness for the numerical solutions based on some regularity conditions. Section 4 is devoted to derive the optimal error estimates for the discrete variational formulation of equations (1)-(3). Finally, some numerical experiments are tested to confirm the established theoretical results and explore the effect of varying stabilized parameters to the errors. In this work, the letter c denotes the general positive constant, which depends on the smallest angle in the triangulation $\mathcal{T}_h$ and domain Ω, independents of the mesh size h and time-step Δt.

2. Preliminaries

2.1. Basic results. In order to present the weak formulation for equations (1)-(3), we need to introduce some Sobolev spaces:

$$X = H_0^1(\Omega)^2, \quad Y = L^2(\Omega)^2, \quad M = L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q dx = 0\}.$$

The spaces $L^2(\Omega)^m$ $(m = 1, 2)$ are endowed with the standard L^2-scalar product $(\cdot, \cdot)$ and norm $\| \cdot \|_0$, the spaces $H_0^1(\Omega)$ and X are equipped with the scalar product $(\nabla u, \nabla v)$ and norm $\|u\|_1$, $\forall\ u, v \in\ H_0^1(\Omega)$ or X.

Let H^{-1} be a dual, with respect to L^2-duality, space to H_0^1 with the corresponding norm:

$$\|f\|_{-1} = \sup_{0 \neq u \in H_0^1} \frac{(f, u)}{\|u\|_1}, \quad f \in H^{-1}.$$

Set

$$Au = -\Delta u, \quad \forall\ u \in D(A) = H^2(\Omega)^2 \cap X.$$

In particular, $D(A^{\frac{1}{2}}) = X,\ D(A^0) = Y$. It is known [3, 13] that

$$\|\ v\ \|_0^2 \leq \gamma_0 \|v\|_1^2, \quad \forall\ v \in X; \quad \|v\|_1^2 \leq \gamma_0 \|Av\|_0^2, \quad \forall\ v \in D(A),$$

where γ_0 is a positive constant only depending on Ω.

We usually make the following assumption about the prescribed data for problem (1)-(3) (see [9, 13]).