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AN ADAPTIVE IMMERSED FINITE ELEMENT METHOD WITH ARBITRARY LAGRANGIAN-EULERIAN SCHEME FOR PARABOLIC EQUATIONS IN TIME VARIABLE DOMAINS

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Abstract. We first propose an adaptive immersed finite element method based on the a posteriori error estimate for solving elliptic equations with non-homogeneous boundary conditions in general Lipschitz domains. The underlying finite element mesh need not fit the boundary of the domain. Optimal a priori error estimate of the proposed immersed finite element method is proved. The immersed finite element method is then used to solve parabolic problems in time variable domains together with an arbitrary Lagrangian-Eulerian (ALE) time discretization scheme. An a posteriori error estimate for the fully discrete immersed finite element method is derived which can be used to adaptively update the time step sizes and finite element meshes at each time step. Numerical experiments are reported to support the theoretical results.

Key words. Immersed finite element, adaptive, a posteriori error estimate, time variable domain.

1. Introduction

Partial differential equations in time variable domains have tremendous interests in scientific and engineering applications including, for example, fluid-structure interaction [4, 18, 16] or melting process [5]. We consider in this paper the following parabolic equations in a time variable domain

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{in } \Omega(t), \text{ a.e. } t \in (0,T), \tag{1}$$

$$u = 0 \quad \text{on } \Gamma(t), \text{ a.e. } t \in (0,T), \tag{2}$$

$$u = u_0 \quad \text{in } \Omega(0), \tag{3}$$

where $T > 0$ is the length of the time interval, $\Omega(t) \subset \mathbb{R}^2$ is a bounded domain at time t with Lipschitz boundary $\Gamma(t)$. We remark that the results in this paper can be easily extended to deal with problems with non-homogeneous Dirichlet boundary condition and other types of boundary conditions such as Neumann or Robin conditions.

Let $F_t : \hat{\Omega} \to \Omega(t)$ be the bijective map which for any $t \in (0,T)$, maps the reference domain $\hat{\Omega}$ to $\Omega(t)$. The problem (1)-(3) will be discretized in time by the following arbitrary Lagrangian-Eulerian (ALE) scheme [16, 17] (see also section 3 below):

$$\frac{U^n - \bar{U}^{n-1}}{\tau_n} - \mathbf{v}^n \cdot \nabla U^n - \Delta U^n = f^n \quad \text{in } \Omega^n = \Omega(t^n), \tag{4}$$

where $\bar{U}^{n-1}$ and $\mathbf{v}^n$ are defined by

$$\bar{U}^{n-1} = U^{n-1}(F_{t^{n-1}}(F_{t^n}^{-1}(x))), \quad \mathbf{v}^n = (\partial_t F_t)|_{t=t^n}(F_{t^n}^{-1}(x)), \quad \forall x \in \Omega^n.$$

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One approach is to solve (4) by the finite element method using the mesh which is the map of a fixed finite element mesh in the reference domain $\hat{\Omega}$. This approach has the difficulty of possible mesh distortions which may lead to undesirable remeshing procedures in practical applications. We also remark that the ALE scheme is closely related to the variable mesh method in [20] whose convergence is also studied in [26].

In this paper we propose to solve (4) by using the immersed finite element method in which the finite element meshes need not fit the boundary of the domain. This allows one to combine the technique of adaptive finite element method based on a posteriori error estimates to obtain a fully adaptive algorithm for solving (1)-(3) with error control which achieves quasi-optimal error reduction as solving parabolic equations on time invariant domains (cf. [6]). We remark that immersed finite element or finite difference methods have been extensively studied in the literature. In the finite difference setting, we refer to the immersed boundary method in [28], the immersed interface method in [22, 24], the ghost fluid method in [27], and the references therein. In the finite element framework, we refer to the work of [25, 10] for elliptic problems with discontinuous coefficients in which finite element basis functions are locally modified for elements intersection the interface where the coefficient jumps. In [8] the adaptive immersed interface finite element method based on a posteriori error estimates is proposed for elliptic and Maxwell equations with discontinuous coefficients.

In this paper we first develop an adaptive immersed finite element method based on the a posteriori error estimate for solving elliptic equations with non-homogeneous boundary condition in general Lipschitz domains. We remark that the a posteriori error estimation and adaptive finite element methods for elliptic problems are extensively studied in the literature for polygonal domains with the exception of [12] in which boundary fitted finite element meshes are used. The boundary fitted finite element mesh has the difficulty in refining boundary elements which may destroy the mesh shape regular property if mesh regularization techniques are not used. In this paper we extend the construction of immersed interface finite element in [8] and propose an immersed finite element method to solve elliptic problems on domains with piecewise smooth boundary. Our construction is equivalent to solving the problem on a boundary fitted finite element mesh that satisfies the maximum angle condition. Thus optimal a priori error estimates are guaranteed if the solution are smooth in $H^2(\Omega)$. We also derive a reliable and efficient a posteriori error estimate by introducing a Clément type interpolation operator and using a result of [15] to localize the approximation error of the non-homogeneous boundary condition in $H^{1/2}$ norm. We also refer to the work of [30] and the references therein for the study of a posteriori error estimation for elliptic problems with non-homogeneous boundary conditions in polygonal or polyhedral domains.

We next apply the immersed finite element method for the elliptic problem developed in the first part of this paper to solve the ALE scheme (12) and obtain a fully discrete immersed finite element method for (1)-(3). We derive an a posteriori error estimate of residual type which can be used to adapt the meshes and time step sizes in practical computations. The derived a posteriori error estimate reduces to the standard a posteriori error estimates for parabolic equations in e.g. [29, 7] if the domains are not variable in time. The new difficulty of estimating the parabolic extension of the discrete boundary data on the variable time domain is overcome by using a deep theorem of Verchota in harmonic analysis on the solvability of