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A FINITE ELEMENT DUAL SINGULAR FUNCTION METHOD TO SOLVE THE STOKES EQUATIONS INCLUDING CORNER SINGULARITIES

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Abstract. The finite element dual singular function method [FE-DSFM] has been constructed and analyzed accuracy by Z. Cai and S. Kim to solve the Laplace equation on a polygonal domain with one reentrant corner. In this paper, we impose FE-DSFM to solve the Stokes equations via the mixed finite element method. To do this, we compute the singular and the dual singular functions analytically at a non-convex corner. We prove well-posedness by using the contraction mapping theorem and then estimate errors of the algorithm. We obtain optimal accuracy $O(h)$ for velocity in $\mathbf{H}^1(\Omega)$ and pressure in $L^2(\Omega)$, but we are able to prove only $O(h^{1+\lambda})$ error bounds for velocity in $\mathbf{L}^2(\Omega)$ and stress intensity factor, where λ is the eigenvalue (solution of (4)). However, we get optimal accuracy results in numerical experiments.

Key words. Stokes equations, dual singular function method, corner singularity, incompressible fluids.

1. Introduction

Solutions of elliptic boundary value problems on a domain with corners have singular behaviors near the corners. This occurs even when the given data of the governed equations is very smooth. Such singular behavior affects the accuracy of the finite element method throughout the whole domain. In order to overcome the singularity problem, the finite element dual singular function method [FE-DSFM] has been constructed in [3] to solve the Laplace equation and performed numerical tests in [4]. And then it is extended to solve the Helmholtz equation in [9] and the interface problem in [8]. The goal of this paper is to reconstruct FE-DSFM to solve Stokes equations:

$$\begin{aligned} -\mu\triangle\mathbf{u} + \nabla p &= \mathbf{f}, \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u} &= 0, \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0}, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

with $\mathbf{f}$ is a given function in $\mathbf{H}^{-1}(\Omega)$, Ω is a computational domain, and $\mu = Re^{-1}$ is the reciprocal of the Reynolds number. Here the unknowns are the (vector) velocity field $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ and the (scalar) pressure $p \in L_0^2(\Omega)$.

If the solution of (1) is smooth enough, namely $(\mathbf{u}, p) \in \mathbf{H}^{s+1}(\Omega) \times H^s(\Omega)$ with $s \geq 1$, and if a suitable finite element pair is imposed for velocity and pressure, then the finite element solution $(\mathbf{u}_h, p_h)$ using the standard mixed method has optimal error bounds as shown in [1, 6]:

$$\|\mathbf{u}-\mathbf{u}_h\|_0 + h\|\mathbf{u}-\mathbf{u}_h\|_1 + h\|p-p_h\|_0 \leq Ch^{s+1}\left(\|\mathbf{u}\|_{s+1} + \|p\|_s\right), \tag{2}$$

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where h is the biggest mesh size. However, if $s < 1$, then the error bounds of the method become only

$$\|\mathbf{u}-\mathbf{u}_h\|_0 + h^s\|\mathbf{u}-\mathbf{u}_h\|_1 + h^s\|p-p_h\|_0 \le Ch^{2s}\left(\|\mathbf{u}\|_{s+1} + \|p\|_s\right). \tag{3}$$

So we call $(\mathbf{u}, p)$ a singular solution for the case $s < 1$, otherwise a regular solution. Because the singularity is due to reentrant corners of computational domain Ω, we assume that Ω is an open and bounded polygonal domain in $\mathbb{R}^2$ with one reentrant corner. Extension to the domain with a finite number of reentrant corners is straightforward.

Let ω be the internal angle. Without the loss of generality, we assume that the corresponding vertex is at the origin and that the internal angle ω is spanned by the two half-lines $\theta = 0$ and $\theta = \omega$. We denote Γ_{in} for 2 edges on the boundary including the reentrant corner and Γ_{out} for other parts of the boundary. Even though the singular functions are already computed in [10], we will derive those again in §6 to get more advanced properties of the singular functions and newly find out the dual singular functions in (8) below.

The singular function $(\mathbf{u}_s, p_s)$, where $\mathbf{u}_s = (u_s, v_s)$, can be summarized with the eigenvalue $\lambda(> 0)$ which is the solution of

$$\sin^2(\lambda\omega) = \lambda^2\sin^2(\omega), \tag{4}$$

by

$$\begin{pmatrix} u_d \\ v_d \\ p_d \end{pmatrix} = d_1 \begin{pmatrix} -r^{-\lambda}\dfrac{\lambda}{\mu}\sin(\theta)\sin((1+\lambda)\theta) \\ -r^{-\lambda}\dfrac{1}{\mu}\left(\sin(\lambda\theta) - \lambda\sin(\theta)\cos((1+\lambda)\theta)\right) \\ 2r^{-\lambda-1}\lambda\cos((1+\lambda)\theta) \end{pmatrix} + d_2 \begin{pmatrix} r^{-\lambda}\dfrac{1}{\mu}\left(\sin(\lambda\theta) + \lambda\sin(\theta)\cos((1+\lambda)\theta)\right) \\ r^{-\lambda}\dfrac{\lambda}{\mu}\sin(\theta)\sin((1+\lambda)\theta) \\ 2r^{-\lambda-1}\lambda\sin((1+\lambda)\theta) \end{pmatrix}, \tag{5}$$

where

$$C_1 = \sin(\lambda\omega) + \lambda\sin(\omega)\cos((1-\lambda)\omega) \quad \text{and} \quad C_2 = \lambda\sin(\omega)\sin((1-\lambda)\omega).$$

We note that the singular function $(\mathbf{u}_s, p_s)$ is the solution of homogeneous Stokes equations with vanishing Dirichlet boundary condition at Γ_{in}. And λ has to be a positive real number and $(\mathbf{u}_s, p_s) \in \mathbf{H}^{1+\lambda}(\Omega) \times H^{\lambda}(\Omega)$. As the conclusion in Lemma 6.1 below, $\lambda = 1$ for any $\omega \le \pi$, so $(\mathbf{u}_s, p_s) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$ is a regular solution and it becomes a singular solution for the case $\lambda < 1$, namely $\omega > \pi$. Moreover (4) has a unique non-trivial solution $\lambda \in \mathbb{R}$ for the case $\pi < \omega \le \beta\pi$. where $\beta :\approx 1.430296653124203$. And (4) has 2 non-trivial real solutions $0.5 < \lambda_1 < \lambda_2 < 1$ for the case $\omega \in (\beta\pi, 2\pi)$. In addition, $\lambda = 0.5$ is the unique non-trivial solution for $\omega = 2\pi$.

Let η be a smooth cut-off function which is equal one identically in neighborhood of origin, and the support of η is small enough so that the functions $\eta\mathbf{u}_s$ vanishes identically on $\partial\Omega$. Then, in general, the solution $(\mathbf{u}, p)$ including singular parts of