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# VARIATIONAL MULTISCALE A POSTERIORI ERROR ESTIMATION FOR $2^{nd}$ AND $4^{th}$-ORDER ODES

DIEGO IRISARRI AND GUILLERMO HAUKE

**Abstract.** In this paper, an explicit a posteriori error estimator is developed for second and fourth order ODEs solved with the Galerkin method that, remarkably, provides exact pointwise error estimates. The error estimator is derived from the variational multiscale theory, in which the subgrid scales are approximated making use of fine-scale Green's functions. This methodology can be extended to any element type and order. Second and fourth order differential equations cover a great variety of problems in mechanics. Two examples with application in elasticity have been studied: the axially loaded beam and the Euler-Bernoulli beam. Because the error estimator is explicit, it can be very easily implemented and its computational cost is very small. Apart from pointwise error estimates, we present local and global a posteriori error estimates in the $L^1$-norm, the $L^2$-norm and the $H^1$-seminorm. Finally, convergence rates of the error and the efficiencies of the estimator are analyzed.

**Key words.** a posteriori error estimation, 1D linear elasticity, Euler-Bernoulli beam, pointwise error, variational multiscale theory.

## 1. Introduction

Second and $4^{\text{th}}$-order ODEs and, in general, elliptic differential equations have been thoroughly studied using finite element methods (FEM). Specially, the standard Galerkin method gives rise to satisfactory solutions for these types of equations, helping the FEM achieve a widespread use by scientist and engineers. However, it is well-known that numerical methods have an inherent error that, basically, depends on the discretization and the order of the numerical method. Accordingly, in order to evaluate the quality of the FEM solution, it is convenient to quantify the numerical error which is committed. Furthermore, a posteriori error estimation can be exploited by adaptive methods to reduce the error where it is more beneficial.

There exists a broad literature on a posteriori error estimation for FEM which can be classified in three groups [2]:

(1) *Residual-based methods*. The proposed estimator belongs to this category. They are also called *explicit methods* since the error estimate is based only on the information provided by the FEM solution. The error is computed via interior residuals or/and inter-element residuals. They were proposed for the first time by Babuška et al. [4, 5].
(2) *Recovery-based methods*. Zienkiewicz and Zhu [38] developed these techniques, which take advantage of superconvergent properties of the solution. Satisfactory results are achieved for a wide variety of problems. The main idea consists of estimating the error by comparing a smoothed gradient with the gradient of the FEM solution. A general background might be found in [1].

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(3) *Auxiliary-problem-based methods*. Lately, important advances have been made in this category in which the error is estimated by solving new differential equations. For this reason, they also receive the name of *implicit methods*. They involve the calculation of residuals of the FEM solution. Typically, these differential equations are applied to a subdomain (an element or a patch of elements). Many researchers have worked in this matter [6, 3]. Ladèveze et al. proposed a local error estimator based on the recovery of equilibrated fluxes [28, 29]. Applications to linear elasticity have been made recently by Verfürth et al. [31], Carstensen et al. [8], Parés, Díez, Huerta and co-workers [35, 9] and Masud et al. [32], where nearly incompressible elasticity is studied. Goal-oriented error estimators have been developed, see Oden et al. [34, 37], in which the dual problem is solved and it involves the computation of influence functions in order to relate the residuals to specific quantities of interest.

In this work, the error estimator is based on the variational multiscale theory (VMS) [24, 26], in which the solution is split into resolved and unresolved scales. Precisely, the unresolved scales present a paradigm from which the error of the finite element solution can be calculated or estimated. In the theory, the interior residuals and inter-element jumps emerge naturally as error sources.

Previous works of the group on this technology were devoted to the transport equation solved with stabilized methods [18, 19, 21, 17, 20, 15, 22]. Here, this technology is extended to second and fourth order differential equations which can be solved with the Galerkin method. Furthermore, following [27] the fine-scale Green's functions have been numerically computed and exploited to obtain expressions for the local and global errors. This procedure has been clarified in the Appendix, so it can be extended to other equations and any element type.

Following the variational multiscale theory, a few relevant articles have studied the fine scales, or unresolved scales, [11, 10, 27, 30], revealing that under most circumstances they are nearly local. That is, for certain class of methods the error is mostly confined inside the element. This is an important property that has been exploited in this work to calculate the error in each element.

Also, beyond existing work on VMS error estimation, pointwise error estimates are studied in this paper. This field has been treated previously for other authors in elliptic problems such as Nochetto [33] using regularized Green's functions. A prominent work was carried out by Prudhomme et al. [36] where quantities of interest of the error are measured and tested in one-dimensional problems. However, the present theory provides a simple way to attain an exact representation of the pointwise error.

Following the introduction, we present in Sec. 2 the background of *VMS error estimation*. The split of coarse and fine-scale spaces is discussed and it is shown that the error can be assessed using the fine-scale variational form. At the end of the section, the general expressions for the pointwise error estimator and for a domain are established. In Sec. 3 and 4, we address the error estimation for the 1-D axially loaded beam and the Euler-Bernoulli beam problem, respectively. In both sections, numerical examples illustrate the behavior of the error estimator. In Section 5, we explain how this estimator can be extended to multi-dimensional problems. Finally, we remark the conclusions of this work.

## 2. The VMS error estimation framework

### 2.1. FEM formulation.