INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 12, Number 2, Pages 366–383

SPECTRAL APPROXIMATION OF TIME-HARMONIC MAXWELL EQUATIONS IN THREE-DIMENSIONAL EXTERIOR DOMAINS

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Abstract. We develop in this paper an efficient and robust spectral-Galerkin method for solving the three-dimensional time-harmonic Maxwell equations in exterior domains. We first reduce the problem to a bounded domain by using the capacity operator which characterizes the transparent boundary condition (TBC). Then, we adopt the transformed field expansion (TFE) approach to reduce the problem to a sequence of Maxwell equations in a spherical shell. Finally, we develop an efficient spectral algorithm by using Legendre approximation in the radial direction and vector spherical harmonic expansion in the tangential directions.

Key words. Maxwell equations, exterior problems, transparent boundary conditions, vector spherical harmonics, Legendre spectral method.

1. Introduction

We consider in this paper the approximation of the time-harmonic Maxwell equations in a three-dimensional exterior domain:

$$
\begin{aligned}
&-\mathrm{i}\omega\mu\boldsymbol{H}+\mathbf{curl}\,\boldsymbol{E}=\mathbf{0},\quad -\mathrm{i}\omega\varepsilon\boldsymbol{E}-\mathbf{curl}\,\boldsymbol{H}=\mathbf{0},\quad \text{in}\quad \mathbb{R}^3\backslash\bar{D};\\
&\boldsymbol{E}\times\boldsymbol{n}|_{\partial D}=\boldsymbol{g};\quad \lim_{r\to\infty} r\big(\sqrt{\mu/\varepsilon}\,\boldsymbol{H}\times\boldsymbol{e}_r-\boldsymbol{E}\big)=0,
\end{aligned}
\tag{1}
$$

where D is a three-dimensional, simply connected, bounded scatterer, $\mathrm{i}=\sqrt{-1}$ is the complex unit, $\boldsymbol{g}$ is resulted from a given incident field, μ is the magnetic permeability, ε is the electric permittivity, ω is the frequency of the harmonic wave, $\boldsymbol{n}$ is the unit outward normal of D and $\boldsymbol{e}_r=\boldsymbol{x}/r$ with $r=|\boldsymbol{x}|$. The boundary condition at infinity in (1) is known as the Silver-Müller radiation condition.

The Maxwell equations (1) play an important role in many scientific and engineering applications, and are also of fundamental mathematical interest (see e.g., [13, 4, 11]). Despite its seemingly simplicity, the system (1) is notoriously difficult to solve numerically. Some of the main challenges include: (i) the indefiniteness when ω is not small; (ii) highly oscillatory solutions when ω is large; (iii) the incompressibility (i.e., $\mathrm{div}(\mu\boldsymbol{H})=\mathrm{div}(\varepsilon\boldsymbol{E})=0$), which is implicitly implied by (1); and (iv) the unboundedness of the domain. On the one hand, one needs to construct approximation spaces such that the discrete problems are well posed and lead to good approximations for a wide range of wave number. On the other hand, a perhaps more difficult problem is to develop efficient algorithms for solving the indefinite linear system, particularly for large wave numbers, from the given discretization. We refer to [11] and the references therein, for various contributions with respect to numerical approximations of the time-harmonic Maxwell equations. Most notably,

Received by the editors November 10, 2014.
1991 *Mathematics Subject Classification.* 65N35, 65N22, 65F05, 35J05.
The first two authors are supported by NFS grant DMS-1215066 and AFOSR FA9550-11-1-0328. The third author is partially supported by Singapore MOE AcRF Tier 1 Grant (RG 15/12), MOE AcRF Tier 2 Grant (MOE 2013-T2-1-095, ARC 44/13) and Singapore A*STAR-SERC-PSF Grant (122-PSF-007).

a very popular and effective method for dealing with the unboundedness of the domain is to introduce a perfectly matched layer (PML), initially proposed in [3].

In this paper, we propose a spectral approximation based on the tensor-product of vector spherical harmonics (VSH), which forms a complete orthogonal basis for L^2-vector-valued functions on the spherical surface, and Legendre polynomials in the radial direction. It is well-known that the Maxwell equations with constant magnetic permeability and electric permittivity are separable if D is a ball, and its solution can be explicitly expressed in terms of the VSH and the spherical Hankel functions [13]. While the explicit solution is very useful for some theoretical considerations, it has much less value in practice, since most practical problems would have one or more of the following situations: non-spherical domains, non-constant magnetic permeability and electric permittivity, non-homogeneous source etc., where an explicit solution would not be available.

In order to deal with more general scatterers D and non-homogeneous source functions, we adapt the so-called transformed field expansion (TFE) [15], which has proven to be effective for a variety of situations (cf. [14, 5, 6, 9]). The TFE approach consists of four steps: (i) reduce the problem in an unbounded domain to a bounded domain with transparent boundary conditions; (ii) transform the reduced bounded domain to a separable domain, consider the reduced domain as a perturbation of the separable domain, and expand the solution in term of the perturbation parameter ε; (iii) solve for each expansion coefficient in the separable domain; and (iv) sum up the expansion terms using a robust Padé approximation. The essential step in the above TFE approach is the step (iii), i.e., solve the Maxwell equations in the separable domain (which is a spherical shell in this case) with non-homogeneous source term and non-local boundary conditions at the outer spherical surface.

In this paper, we shall develop an efficient and robust spectral solver for the non-homogeneous Maxwell equations in a spherical shell. More precisely, we shall use VSH to decouple the problem into a sequence of one-dimensional problems that can be efficiently solved using a direct spectral-Galerkin method. Therefore, the entire TFE approach does not involve any iterative solver, and it is robust for low to moderately high wave numbers and to scatterers which have sufficiently smooth boundaries.

The rest of the paper is organized as follows. In the next section, we introduce the VSH and present the formulation of the capacity operator characterizing the exact non-reflecting boundary condition. In Section 3, we present the TFE algorithm, and and formulae in Appendix B. In Section 4, we describe the Legendre spectral-Galerkin method for the reduced one-dimensional problems, and give the numerical results in Section 5. In Appendix A, we provide some useful formulae for the VSH, while in Appendix B, we derive the Maxwell equation in the transformed coordinates, and the recursion formulae in the TFE approach.

2. Vector spherical harmonics and the capacity operator

In this section, we recall some essential properties of VSH, and derive the explicit formula for the capacity operator expressed in terms of VSH, which characterizes the exact DtN boundary condition at the outer spherical surface.

2.1. Vector spherical harmonics. Several versions of VSH with different notation and properties have been used in practice (see e.g., [12, 10, 2, 13, 8, 7]). In what follows, we adopt the family of VSH in [10, 13], and remark its relation with several other families documented in the above literature (see Remark 2.1 below).