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DISCRETE LEAST SQUARES HYBRID APPROXIMATION WITH REGULARIZATION ON THE TWO-SPHERE

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Abstract. In this paper we consider the discrete constrained least squares problem coming from numerical approximation by hybrid scheme on the sphere, which applies both radial basis functions and spherical polynomials. We propose a novel $l_2 - l_1$ regularized least square model for this problem and show that it is a generalized model of the classical "saddle point" model. We apply the alternating direction algorithm to solve the $l_2 - l_1$ model and propose a convenient stopping criterion for the algorithm. Numerical results show that our model is more efficient and accurate than other models.

Key words. Regularized least squares, hybrid approximation, alternating direction method.

1. Introduction

Numerical approximation on the sphere is nowadays a widely studied problem arising in plenty of science landscapes such as geophysics, astrophysics, and surface reconstruction. Amongst varieties of different approaches, the hybrid approximation scheme [7, 11, 16, 19] seems an attractive method, which employs both the radial basis functions (RBF) and spherical harmonic polynomials. Often the underlying motivation has been the need to approximate geophysical quantities. It is well understood that the radial basis functions could approximate rapidly varying data over short distance effectively, whereas the spherical harmonic polynomials are more suitable for slowly varying data on a global scale.

In this paper we will discuss approximating a continuous function $f \in C(\mathbb{S}^2)$ using both radial basis functions and spherical harmonic polynomials, where $\mathbb{S}^2$ represents the unit sphere in three dimensional space as

$$\mathbb{S}^2 = \{\mathbf{x} = (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\},$$

and $C(\mathbb{S}^2)$ denotes the space of all continuous functions defined on $\mathbb{S}^2$. We assume that the values of f are given at a distinct data point set

$$X_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N,\ N \in \mathbb{N}\} \subset \mathbb{S}^2.$$

To construct the radial basis functions, we choose all points in X_N as the center points and employs a (strictly) positive definite kernel ϕ [15, 19, 20] which satisfies

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \phi(\mathbf{x}_i, \mathbf{x}_j)\alpha_j \geq 0, \tag{1}$$

for any point set of $X_N \subset \mathbb{S}^2$ and for all $N \in \mathbb{N}$, with equality for distinct points $\mathbf{x}_j$ only if $\alpha_1 = \alpha_2 = \ldots = \alpha_N = 0$. Then the RBFs are defined as $\phi(\cdot, \mathbf{x}_j)$ with $j = 1, \ldots, N$. Additionally, we assume that kernel ϕ is zonal, which means

$$\phi(\mathbf{x}_i, \mathbf{x}_j) = \phi(\mathbf{x}_i \cdot \mathbf{x}_j),$$

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for arbitrary $i, j = 1, \ldots, N$, where $\mathbf{x}_i \cdot \mathbf{x}_j$ denotes the Euclidean inner product in $\mathbb{R}^3$. Then we can define a space of RBFs as

$$\mathcal{X}_{X_N,\phi} = \mathcal{X}_N = \text{span}\{\phi(\cdot, \mathbf{x}_j) : \ \mathbf{x}_j \in X_N\}.$$

Further more, denote by

$$\mathcal{F}_\phi = \text{span}\{\phi(\cdot, \mathbf{x}_j) : \ \mathbf{x}_j \in \mathbb{S}^2, j \in 1, \ldots, N, \ N \in \mathbb{N}\},$$

which is a reproducing kernel pre-Hilbert space [19] under the inner product

$$\left\langle \sum_{i=1}^{N} \alpha_i \phi(\cdot, \mathbf{x}_i), \sum_{j=1}^{N} \alpha'_j \phi(\cdot, \mathbf{x}_j) \right\rangle_\phi = \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha'_j \phi(\mathbf{x}_i, \mathbf{x}_j), \tag{2}$$

and the norm

$$\left\| \sum_{i=1}^{N} \alpha_i \phi(\cdot, \mathbf{x}_i) \right\|_\phi = \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \phi(\mathbf{x}_i, \mathbf{x}_j), \tag{3}$$

with $\alpha_j \in \mathbb{R}, \ j = 1, \cdots, N$. Let $\mathcal{N}_\phi$ be the completion of $\mathcal{F}_\phi$ and then we can obtain that $\mathcal{N}_\phi$ is a reproducing kernel Hilbert space (RKHS). It is well known and could be easily verified that the reproducing kernel [11] ϕ of $\mathcal{N}_\phi$ satisfies

$$\phi(\mathbf{x}, \mathbf{y}) = \phi(\mathbf{y}, \mathbf{x}), \quad \forall \ \mathbf{x}, \ \mathbf{y} \in \mathbb{S}^2,$$

$$\phi(\cdot, \mathbf{y}) \in \mathcal{N}_\phi, \qquad \mathbf{y} \in \mathbb{S}^2,$$

and

$$\langle f, \phi(\cdot, \mathbf{y}) \rangle_\phi = f(\mathbf{y}), \quad \mathbf{y} \in \mathbb{S}^2, \quad f \in \mathcal{N}_\phi.$$

Another space we will apply is defined as

$$\begin{aligned} \mathbb{P}_L &= \{\textit{spherical polynomials of degree} \ \leq L\} \\ &= \text{span}\{Y_{\ell,k} : \ell = 0, \ldots, L, k = 1, \ldots, 2\ell + 1\}, \end{aligned}$$

with its dimension denoted by

$$d_L = \sum_{\ell=0}^{L} (2\ell + 1) = (L + 1)^2.$$

Here $Y_{\ell,k}$ is a fixed $\mathbb{L}_2$-orthonormal real spherical harmonic polynomial [1] of degree ℓ and order k defined on $\mathbb{S}^2$, which we can express by the denotation of $\mathbb{L}_2(\mathbb{S}^2)$ inner product on $\mathbb{S}^2$ as

$$\langle Y_{\ell,k}, Y_{\ell',k'} \rangle_{\mathbb{L}_2} = \int_{\mathbb{S}^2} Y_{\ell,k} Y_{\ell',k'} d\omega(\mathbf{x}) = \delta_{\ell,\ell'} \delta_{k,k'},$$

with

$$\ell, \ell' = 0, \ldots, L, \ k = 1, \ldots, 2\ell + 1, k' = 1, \ldots, 2\ell' + 1,$$

where $d\omega(\mathbf{x})$ denotes the normalized surface measure, and $\delta_{\ell,\ell'}$ is the Kronecker delta. According to the addition theorem, a zonal radial basis function has a expansion of the form

$$\phi(\cdot, \mathbf{x}) = \sum_{\ell=0}^{\infty} \hat{\phi}_\ell P_\ell(\cdot, \mathbf{x}) = \sum_{\ell=0}^{\infty} \frac{\hat{\phi}_\ell}{2\ell + 1} \sum_{k=1}^{2\ell+1} Y_{\ell,k}(\cdot) Y_{\ell,k}(\mathbf{x}), \tag{4}$$

where $\hat{\phi}_\ell > 0, \ \ell = 0, \ldots, \infty$ when ϕ is a strictly positive kernel. Here P_ℓ is the Legendre polynomial of degree ℓ in 3-dimension normalized to $P_\ell(1) = 1$.