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A WEIGHTED VARIATIONAL FORMULATION BASED ON PLANE WAVE BASIS FOR DISCRETIZATION OF HELMHOLTZ EQUATIONS

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Abstract. In this paper we are concerned with numerical methods for solving Helmholtz equations. We propose a new variant of the Variational Theory of Complex Rays (VTCR) method introduced in [15, 16]. The approximate solution generated by the new variant has higher accuracy than that generated by the original VTCR method. Moreover, the accuracy of the resulting approximate solution can be further increased by adding two suitable positive relaxation parameters into the new variational formula. Besides, a simple domain decomposition preconditioner is introduced for the system generated by the proposed variational formula. Numerical results confirm the efficiency of the new method.

Key words. Helmholtz equations, wave basis functions, variational formulation, error estimate, preconditioner, iteration counts.

1. Introduction

In recent years, the study of the vibrational behavior of mechanical systems has become a cornerstone of the design of industrial products and of the optimization of their performances. A key point in structural design is the modeling and calculation of the vibrational response of industrial structures. Strategies to analyze the structural and acoustical behavior of structures have been developed based on the finite element method [5, 17, 23, 1, 22], the boundary element method [10, 2, 4]. However, these methods are limited mainly to low-frequency problems and are either inaccurate or costly.

Today, there are also dedicated computational strategies for the resolution of medium-frequency problems, known as Trefftz methods [25], which differ from the traditional FEM and the BEM in the sense that the basis functions in Trefftz methods are chosen as some exact solutions of the governing differential equation without boundary condition. These approaches include the Ultra Weak Variational Formulation (UWVF) (see [3, 7]), the plane wave least-squares method [19], the plane wave discontinuous Galerkin methods (PWDG) (see [8, 11]), the discontinuous enrichment method [6] and the Variational Theory of Complex Rays (VTCR) introduced in [15, 16, 20] (see also [14] and [21]). An important advantage of these approaches is that they are capable of producing an approximate solution with high accuracy by using only a small number of DOFs. In this paper, we are interested in the development of the VTCR method.

The VTCR method has some similarity with the UWVF method. There are two basic ingredients in the both methods: a triangulation on the underlying domain and a set of wave basis functions in each element. But, two different kinds of unknown functions are chosen in the variational equations of these two methods.

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For the VTCR method, the restrictions of the desired approximate solution on every elements are chosen as the unknown functions, and some weak continuity of the traces of the approximation across each local interface generated by the triangulation is imposed by a direct variational formulation involved the traces. For the UWVF method, the Robin boundary functions of the approximate solution on the boundaries of every elements are chosen as the unknown functions, the conjugation of each Robin boundary function has to be defined by introducing an additional mapping. Then, in the UWVF method, the same weak continuity of the traces is imposed by an indirect variational formulation involved the Robin boundary functions and their conjugations. The design of the UWVF method is based on the idea in the non-overlapping domain decomposition with Robin interface conditions, on contrary to this, the design of the VTCR method is only based on an intuitive idea to impose some weak continuity of the traces. It seems that the variational equation in the VTCR method is simpler than the one in the UWVF method, and the VTCR method is easier to implement than the UWVF method. By the way, the unknown functions in the PWDG method is also the restrictions of the desired approximate solution on every elements, and the PWDG method was derived by using the techniques in the DG method.

In this paper, we present a new variant of the VTCR method. In the variational equation of the original VTCR method, two different kinds of traces of the test function and the trial function are used in each integral on the common interface between two neighboring elements. The design of our method is also based on an intuitive idea to impose some weak continuity of the traces, but we change that variational equation such that the same kinds of traces of the test function and the trial function are used in each interface integral. We find that the approximate solution generated by the new variant has higher accuracy than the one generated by the original VTCR method. More importantly, the accuracy of the new approximation can be improved further by adding two suitable relaxation parameters into the new variational equation. For convenience, the resulting variational formulation is called a weighted variational formulation (WVF). We prove a L^2 error estimate of the approximate solution generated by the discrete WVF method. Numerical experiments for both two-dimensional and three-dimensional problems show that the new WVF method is obviously superior to the original VTCR method, and is as good as the UWVF method in convergence (the new WVF method seems easier to implement than the UWVF method). Unlike the existing discretization methods for Helmholtz equations, the coefficient matrix of the algebraic system generated by the WVF method is Hermite positive definite, so the algebraic system is easier to solve.

To solve the algebraic system generated by the WVF method in an efficient manner, we construct a simple domain decomposition preconditioner for the coefficient matrix of the algebraic system. The numerical results indicate that the systems generated by the WVF method for Helmholtz equations can be solved rapidly by the preconditioned GMRES method with the proposed preconditioner.

The paper is organized as follows: In Section 2, we briefly review the Variational Theory of Complex Rays for Helmholtz equations. In Section 3, we present a new variant of the VTCR for Helmholtz equations, with two relaxation parameters. In Section 4, we describe discretization of the variational formulation and derive an error estimate of the resulting approximate solution. In section 5, we construct a domain decomposition preconditioner for the stiffness matrix associated with the