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GEOMETRIC MULTIGRID METHODS ON STRUCTURED TRIANGULAR GRIDS FOR INCOMPRESSIBLE NAVIER-STOKES EQUATIONS AT LOW REYNOLDS NUMBERS

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This paper is dedicated to Francisco Lisbona on occasion of his 65th birthday

Abstract. The main purpose of this work is the efficient implementation of a multigrid algorithm for solving Navier-Stokes problems at low Reynolds numbers in different triangular geometries. In particular, a finite element formulation of the Navier-Stokes equations, using quadratic finite elements for the velocities and linear finite elements to approximate the pressure, is used to solve the problem of flow in a triangular cavity, driven by the uniform motion of one of its side walls. An appropriate multigrid method for this discretization of Navier-Stokes equations is designed, based on a Vanka type smoother. Moreover, the data structure used allows an efficient stencil-based implementation of the method, which permits us to perform simulations with a large number of unknowns with low memory consumption and a relatively low computational cost.

Key words. Multigrid methods, Navier-Stokes equations, Vanka smoother, Cavity problem

1. Introduction

One of the most important aspects in the numerical simulation of the Navier-Stokes equations is the efficient solution of the large sparse systems of equations arising from their discretization. This work is focused on an efficient implementation and the solution by geometric multigrid methods of the incompressible Navier-Stokes equations on structured triangular grids.

It is well-known that multigrid methods [2, 4, 8, 17] are among the fastest algorithms to solve large systems of equations, with small convergence factors which are independent of the space discretization parameter, and achieve optimal computational complexity of $\mathcal{O}(N)$, where N is the number of unknowns of the system. Geometric multigrid methods were initially developed for structured grids. However, in order to deal with relatively complex domains, an efficient implementation of this type of multigrid methods can be done on semi-structured triangular grids, see [6]. As a preliminary step towards this generalization, here we develop a geometric multigrid code suitable for efficiently solving this problem on a structured grid arising in a single triangular domain, which later will be part of the semi-structured grid.

An important step in the analysis of partial differential equations (PDE) problems using finite element methods is the construction of the large sparse matrix $\mathcal{A}$ corresponding to the system of discrete equations. The standard algorithm for computing matrix $\mathcal{A}$ is known as *assembly*, and consists of computing this matrix by iterating over the elements of the mesh and adding from each element of the triangulation the local contribution to the global matrix $\mathcal{A}$. Because of the size of this

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matrix, it is important to store it in an efficient way. However, the data structures needed to represent this type of sparse matrices can cause slowness in the code due to the use of indirect indexing to access the non-zero entries of the matrix. By working on structured grids, the necessary data structures are much more efficient and lead to better performance, due to the fact that explicit assembly of the global matrix is not necessary and that the matrix can be stored using stencils.

In this work, a stencil-based implementation of the Taylor-Hood element for the Navier-Stokes equations is presented, together with the design of an efficient geometric multigrid algorithm, based on a box-type smoother, to solve the large system of equations arising from this type of finite element discretization. More concretely, the outline of this work is as follows. In Section 2, the considered problem is presented, together with the linearization and the proposed finite element discretization. Section 2.1 is devoted to describe the stencil-based implementation of the Taylor-Hood element discretization of Navier-Stokes equations. Section 3 is focused on the design of a suitable geometric multigrid method, based on Vanka-type smoothers. Finally, in Section 4, the lid-driven recirculating flow in a triangular cavity is simulated, using the proposed multigrid solution procedure.

2. Finite element discretization of the Navier-Stokes equations

In this work we consider the Navier-Stokes equations governing a two-dimensional, steady, incompressible flow of constant fluid properties. These equations are written in primitive variables as

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{0}, && \text{in } \Omega, \\ \operatorname{div}\mathbf{u} &= 0, && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g}, && \text{on } \Gamma = \partial\Omega, \end{aligned} \tag{1}$$

where $\mathbf{u} = (u, v)^t$ denotes the velocity vector, p is the pressure, and ν is the kinematic viscosity of the fluid. The Dirichlet boundary condition for the velocity is given by $\mathbf{g}$, which satisfies the following compatibility condition

$$\int_{\partial\Omega} \mathbf{g}\cdot\mathbf{n}\,\mathrm{d}\Gamma = 0, \tag{2}$$

where $\mathbf{n}$ is the outward direction normal to the boundary.

Nonlinear problem (1) is linearized using a fixed point iteration, that is, given a current iterate $(\mathbf{u}^n, p^n)$, in each nonlinear iteration step a problem of the following form has to be solved

$$\begin{aligned} -\nu\Delta\mathbf{u}^{n+1} + (\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1} + \nabla p^{n+1} &= \mathbf{0}, && \text{in } \Omega, \\ \operatorname{div}\mathbf{u}^{n+1} &= 0, && \text{in } \Omega, \\ \mathbf{u}^{n+1} &= \mathbf{g}, && \text{on } \Gamma = \partial\Omega. \end{aligned} \tag{3}$$

Problem (3) is known in the literature as the Oseen problem. We are going to consider its discretization by finite element methods. For this purpose, let $\mathcal{T}_h$ be an admissible triangulation of the domain Ω, that is, Ω is decomposed into a set of triangles $\{K_i\}_{i=1}^N$ in the way that

$$\overline{\Omega} = \bigcup_{i=1}^{N} K_i,$$

and satisfying that the intersection $K_i \cap K_j$, for $i \neq j$, is either empty, a common vertex, or a common edge. Problem (3) is discretized using $P2-P1$ finite elements, where Pk is the space of piecewise polynomial continuous functions of degree k.