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# FAST OPTIMAL $\mathcal{H}_2$ MODEL REDUCTION ALGORITHMS BASED ON GRASSMANN MANIFOLD OPTIMIZATION

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**Abstract.** The optimal $\mathcal{H}_2$ model reduction is an important tool in studying dynamical systems of a large order and their numerical simulation. We formulate the reduction problem as a minimization problem over the Grassmann manifold. This allows us to develop a fast gradient flow algorithm suitable for large-scale optimal $\mathcal{H}_2$ model reduction problems. The proposed algorithm converges globally and the resulting reduced system preserves stability of the original system. Furthermore, based on the fast gradient flow algorithm, we propose a sequentially quadratic approximation algorithm which converges faster and guarantees the global convergence. Numerical examples are presented to demonstrate the approximation accuracy and the computational efficiency of the proposed algorithms.

**Key words.** $\mathcal{H}_2$ approximation, gradient flow, Grassmann manifold, model reduction, MIMO system, stability, large-scale sparse system.

## 1. Introduction

Model reduction, which approximates a linear dynamic system of a higher order by a system of a significantly lower order, received considerable attention in recent years. This problem is important for many applications, such as design of large scale integration chips, analysis and design of micro electro mechanical system devices, weather prediction and control of partial differential equations. There are many existing methods that produce lower-order systems from given high-order systems. Most of these methods fall into two categories. The first category is projection-based methods, such as the Krylov subspace (moment matching) methods, the balanced-truncation method, and proper orthogonal decomposition (POD) methods. The second category is optimization-based methods such as the Hankel optimal model reduction [6] and the $\mathcal{H}_2$ optimal model reduction. For nice reviews of model reduction for large-scale dynamical systems, the readers are referred to [3, 7, 28].

The present paper concerns the optimal $\mathcal{H}_2$ norm model reduction problem, which has been studied by many investigators, see, for instance [4, 9, 13, 15, 22, 23, 24, 25, 26] and the references cited therein. Most of the existing algorithms are not suitable for the reduction of large-scale systems. Algorithms proposed in [9, 15, 22] require solving large-scale Lyapunov equations at each iteration step, making them computationally expensive for large scale systems. Gradient flow algorithm [26] requires computing the exponential of a large-scale matrix at each iteration step which is computationally expensive. Interpolation-based algorithms were proposed in [4, 13, 23, 25] for solving large-scale $\mathcal{H}_2$ optimal model reduction problems. A

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drawback of interpolation-based algorithms is that they often cannot guarantee convergence and the stability of the reduced model.

The main purpose of this paper is to develop fast algorithms suitable for sparse systems of large scales, with the resulting reduced system preserving stability of the original system. Specifically, we treat the optimal $\mathcal{H}_2$ model reduction problem as a minimization problem on the Grassmann manifold. We propose a fast gradient flow algorithm based on the geodesic of the Grassmann manifold which is suitable for large sparse multi-input multi-output (MIMO) systems. The geodesic equation of the Grassmann manifold is easy to compute, which makes it possible to reduce the computational cost. The new algorithm avoids computing the matrix exponential and only involves computation with matrices of small size. We also reformulate the partial derivatives of the cost function to be more computationally effective. The proposed algorithm has nice properties that starting from any initial orthogonal matrix, the iterations remain on the manifold. The convergence of the algorithm is guaranteed. We also derive sufficient conditions for the algorithms to preserve the stability. Based on approximating the cost function by a quadratic function, we propose an algorithm with faster convergence. By combining with the gradient flow method, the global convergence is guaranteed. The two Grassmann manifold-based algorithms proposed in this paper have low computational cost and are suitable for large-scale sparse systems. Since they do not rely on the assumption that the target system has simple poles, they remain robust when the target system has multiple poles.

This paper is organized in eight sections. In Section 2, we introduce the optimal $\mathcal{H}_2$ model reduction problem minimizing over the Grassmann manifold. In Section 3, we describe the gradient flow on the Grassmann manifold for solving the $\mathcal{H}_2$ optimal model reduction problem. Section 4 centers at the development of fast numerical algorithms for the gradient flow on the Grassmann manifold and provides complexity analysis. In Section 5, we compare the proposed fast gradient flow algorithm with the existing gradient flow algorithm. In Section 6, an algorithm with faster convergence is proposed. Section 7 is devoted to a presentation of numerical results of the proposed algorithms. Finally, in Section 8, we draw a conclusion.

## 2. Optimal $\mathcal{H}_2$ Model Reduction

We describe in this section the optimal $\mathcal{H}_2$ model reduction problem.

For given matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{q\times n}$, we consider the linear dynamical systems described by

$$\frac{dx}{dt} = Ax(t) + Bu(t), \tag{1}$$

$$y(t) = Cx(t), \tag{2}$$

where $t \geq 0$ is the time variable, $u \in \mathbb{R}^p$ is the input, $y \in \mathbb{R}^q$ is the output and $x \in \mathbb{R}^n$ is the state of the system. Here, $n$ is the system order, $p$ and $q$ are the number of system inputs and outputs, respectively. The linear system described by equations (1) and (2) is uniquely determined by the state space realization $(A, B, C)$ and the initial condition $x(t_0) = x_0$. In this paper, we assume that the numbers of input and output of the full order system described by equations (1) and (2) are small, that is, $q \ll n$ and $p \ll n$. Moreover, we assume that the matrix $A$ is sparse.

If the system order $n$ is too big, it is not computationally efficient to solve various control problems. We need to construct a reduced order system to approximate the full order system, with preserving certain system properties such as stability and