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NONCONFORMING MIXED FINITE ELEMENT METHODS FOR STATIONARY INCOMPRESSIBLE MAGNETOHYDRODYNAMICS

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Abstract. The main aim of this paper is to study the approximation of nonconforming mixed finite element methods for stationary, incompressible magnetohydrodynamics (MHD) equations in 3D. A family of nonconforming finite elements are taken as the approximation spaces for the velocity field, the piecewise constant element for the pressure and the Nédélec's element with the lowest order for the magnetic field on hexahedra or tetrahedra. A new simple method is adopted to prove the discrete Poincaré-Friedrichs inequality instead of the discrete Helmholtz decomposition method. The existence and uniqueness of the approximate solutions are shown. The convergence analysis is presented and the optimal order error estimates for the pressure in L^2-norm, as well as those in a broken H^1-norm for the velocity field and $H(curl)-$norm for the magnetic field are derived.

Key words. Incompressible MHD equations, Nonconforming mixed finite element method, Optimal error estimates

1. Introduction

Magnetohydrodynamics (MHD) equations is the complicated coupling problem which is composed of electrically conducting fluid and electromagnetic fields. The MHD equations arises in several applications, for example, astronomy and geophysics as well as the associated numerous engineering problems, such as liquid-metal cooling of nuclear reactors, electromagnetic casting of metals, MHD power generation and MHD ion propulsion [1]. Many studies have been already devoted to the incompressible MHD equations. For theoretical results, G. Duvaut and J.-L. Lions [2] first established the existence and uniqueness results for weak and strong solutions of the MHD equations. M. Sermange and R. Temam [3] then analyzed the large time behavior, the regularity properties and bound on the solutions to the MHD equations which are valid for all time. On one hand, a considerable amount of research activity has been devoted to the analysis of numerical methods for the simulation of MHD flows by using finite difference methods (FDMs) [4]-[7]. On the other hand, most of the numerical solutions of the MHD equations are performed with the finite element methods (FEMs) [8]-[12], [14]-[22].

More precisely, in [8]-[12], the studies required that the magnetic field belongs to $H^1(\Omega)^3$. However, in the presence of reentrant corners or edges, setting the magnetic unknowns of the incompressible MHD equations in $H^1(\Omega)^3$ leads to a well-posed problem where the magnetic field cannot be correctly approximated because the magnetic field may have regularity below $H^1(\Omega)^3$ [13]. In order to overcome this difficulty, M. Costabel and M. Dauge [13] first presented a method of regularizing

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time harmonic Maxwell equations where the magnetic field belongs to $H(curl;\Omega)$. This observation recently motivated the work [14], where a mixed formulation of the stationary incompressible MHD problem based on $H(curl)-$conforming (edge) elements to approximate the magnetic field was proposed, and the convergence to weak solutions of the discrete problem was proved. A different strategy to achieve convergence in general polyhedral domains was realized in [15], where an interior penalty discontinuous Galerkin method for a linearized incompressible MHD model problem was applied. Based on the mixed formulation introduced in [14], there have been numerical analysis for schemes available, whose convergence was shown either in the context of existing weak or strong solutions [16]-[26]. A recent summary of known results for the MHD equations, including modeling, analysis, and numerics is [1]. But, almost all the analysis in (FEMs) [8]-[12], [14]-[1] are about the conforming FEMs except [16]. To the best of our knowledge, until now there are few papers focusing on the nonconforming finite element methods (NFEMs).

It is well known that NFEMs play an important role in the numerical approximation of partial differential equations. Firstly, NFEMs have been used effectively especially in fluid and solid mechanics when conforming FEMs and others seem too costly or unstable. Secondly, the mixed finite element approximation to MHD equations needs the stability and the compatibility between the velocity and the pressure finite element spaces satisfying the discrete inf-sup condition [27], NFEMs are much easier to be constructed to satisfy the above condition than conforming FEMs. Thirdly, for some Crouzeix-Raviart type finite elements with the degrees of freedom defined on the edges (or faces) of element or element itself, since the unknowns are associated with the element faces, each degree of freedom belongs to at most two elements, the use of the nonconforming finite elements facilitates the exchange of information across each subdomain and provides spectral radius estimates for the iterative domain decomposition operator [28]. Furthermore, NFEMs for the resolution of a wide range of linear and nonlinear boundary value problems have a great development in the last years [29]-[40]. The authors [41] also proposed a family of low-order nonconforming mixed FEMs to approximate stationary MHD equations and obtained the optimal error estimates in convex polyhedral domains, or domains with a boundary $C^{1,1}$.

As an attempt, we are concerned with NFEMs for nonlinear, fully coupled stationary incompressible MHD equations by the mixed formulation in general Lipschitz polyhedra. We will adopt a family of nonconforming finite elements as approximation space for the velocity field, the piecewise constant element for the pressure and the first kind Nédélec's elements on tetrahedra or hexahedra with the lowest order for the magnetic field. A new method is introduced to prove the discrete Poincar*é*-Friedrichs inequality, which is much easier than the methods used in [27, 42, 43]. Finally, we will show the existence and uniqueness of the approximate solutions and obtain the optimal order error estimates.

This paper is organized as follows: In Section 2, we introduce the variational formulation for the MHD equations. Section 3 will give the nonconforming finite element spaces. In Section 4, we state some important lemmas and prove the existence and uniqueness of discrete solutions. Section 5 will present the convergence analysis and derive the optimal order error estimates.

In this paper, we will use the notations $\|\cdot\|_l, \|\cdot\|_{l,K}$ for $H^l(\Omega)^3, H^l(K)^3$-norm, $|\cdot|_m, |\cdot|_{m,K}$ for $H^m(\Omega)^3, H^m(K)^3$-seminorm, where $H^0(\Omega)^3 = L^2(\Omega)^3$ and $H^0(K)^n = L^2(K)^3, l \geq 0, m \geq 0$ are integer numbers. Throughout the paper, C indicates a positive constant, possibly different at different occurrences, which is