INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 10, Number 4, Pages 815–825

AN EFFICIENT COLLOCATION METHOD FOR A NON-LOCAL DIFFUSION MODEL

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Abstract. The non-local diffusion model provides an appropriate description of the deformation of a continuous body involving discontinuities or other singularities, which cannot be described properly by classical theory of solid mechanics. However, because the non-local nature of the non-local diffusion operator, the numerical methods for non-local diffusion model generate dense or even full stiffness matrices. A direct solver typically requires $O(N^3)$ of operations and $O(N^2)$ of memory where N is the number of unknowns. We develop a fast collocation method for the non-local diffusion model which has the following features: (i) It reduces the computational cost from $O(N^3)$ to $O(N \log^2 N)$ and memory requirement from $O(N^2)$ to $O(N)$. (ii) It requires only one-fold integration in the evaluation of the stiffness matrix. Numerical experiments show the utility of the method.

Key words. collocation method, dense matrices, fast methods, non-local diffusion, peridynamics.

1. Introduction

The classical theory of solid mechanics assumes that all internal forces act through zeros distance. The corresponding mathematical models are described by partial differential equations, which do not provide a proper description of problems with spontaneous formation of discontinuities or other singularities. The non-local diffusion model was proposed as a reformation of solid mechanics [8], which does not explicitly involve the notion of deformation gradients.

Galerkin finite element methods were previously developed and analyzed for the non-local diffusion model [3, 5, 7, 10]. However, these methods face two challenges:

(1) Because of the non-local nature of the non-local diffusion operator, Galerkin finite element methods generate dense or even full matrices. The direct solvers used in solving the resulting discrete systems often require $O(N^3)$ of computational work and $O(N^2)$ of memory where N is the number of degree of freedoms, which are significantly more expensive than the Galerkin finite element methods for the classical models described by differential equations.
(2) Each entry in the stiffness matrix involves two-folds of integration, which makes the evaluation of the stiffness matrix more expensive. Further, a fast solution method can be developed only for a uniform mesh.

In this paper we develop a fast collocation method for the non-local diffusion model. The method has the following features:

(1) The fast method can be developed on both a uniform mesh and a geometrically decreasing mesh. In particular, the latter is particularly suited for problems with singularities. For both meshes, the fast collocation method reduces the computational cost from $O(N^3)$ to $O(N \log^2 N)$ and memory requirement from $O(N^2)$ to $O(N)$.

Received by the editors June 19, 2012, and in revised form, November 11, 2012.
2000 *Mathematics Subject Classification.* 65J, 65R 65N.
This work was supported in part by the National Science Foundation under Grants EAR-0934747 and DMS-1216923 and by the National Natural Science Foundation of China under Grants 91130010 and 11271232.

(2) Only one-fold of integration is needed in the evaluation of the stiffness matrix, which further reduces the computational cost.

The rest of the paper is organized as follows. In §2 we present a non-local diffusion model and review its Galerkin finite element approximations. In §3 we develop a collocation method for the non-local diffusion model. In §4 we develop a fast collocation method on a uniform mesh. In §5 we develop a fast collocation method on a geometrically decreasing mesh. In §6 we conduct numerical experiments to investigate the computational benefits of the fast methods.

2. A non-local diffusion model and its Galerkin finite element approximation

In this section we briefly discuss the non-local diffusion model and its Galerkin finite element approximation.

2.1. A non-local diffusion model. A linear steady-state non-local diffusion model for microelastic materials on a finite bar is given by the following pseudo-differential equation [7, 8]

$$\int_\alpha^\beta \frac{u(x)-u(y)}{|x-y|^r}dy = b(x), \quad x \in (\alpha,\beta). \tag{1}$$

Here $b(x)$ represents the prescribed forcing term and $u(x)$ presents the displacement of the material. $r \geq 0$ is a parameter that characterizes the influence or decaying property of the kernel function.

By the symmetry of x and y the bilinear form $a(u,v)$ defined by

$$a(u,v) := \int_\alpha^\beta v(x) \int_\alpha^\beta \frac{u(x)-u(y)}{|x-y|^r}dydx \tag{2}$$

can be rewritten as

$$\begin{aligned} a(u,v) &= \int_\alpha^\beta \int_\alpha^\beta \frac{v(x)(u(x)-u(y))}{|x-y|^r}dydx \\ &= \int_\alpha^\beta \int_\alpha^\beta \frac{v(y)(u(y)-u(x))}{|x-y|^r}dydx. \end{aligned} \tag{3}$$

which concludes that

$$\int_\alpha^\beta \int_\alpha^\beta \left(\frac{v(x)\big(u(x)-u(y)\big) - v(y)\big(u(y)-u(x)\big)}{|x-y|^r} \right) dydx = 0. \tag{4}$$

The numerator of the integrand in (4) can be decomposed as

$$\begin{aligned} &v(x)\big(u(x)-u(y)\big) - v(y)\big(u(y)-u(x)\big) \\ &\quad = v(x)\big[\big(u(x)-u(y)\big) - \big(u(y)-u(x)\big)\big] \\ &\qquad -\big(v(y)-v(x)\big)\big(u(y)-u(x)\big) \\ &\quad = 2v(x)\big(u(x)-u(y)\big) - \big(v(y)-v(x)\big)\big(u(y)-u(x)\big). \end{aligned} \tag{5}$$

We incorporate (5) into (4) to derive an alternative expression for $a(u,v)$

$$a(u,v) = \int_\alpha^\beta \int_\alpha^\beta \frac{(u(x)-u(y))(v(x)-v(y))}{2|x-y|^r}dydx. \tag{6}$$

The following theoretical results were proved previously [7, 10]: If $r < 1$, i.e., the kernel is integrable, then $a(u,v)$ is a (semi-) positive-definite and bounded bilinear form on $L^2(\alpha,\beta) \times L^2(\alpha,\beta)$. If $r = 1 + 2s$ with $s > 0$, then $a(u,v)$ is a (semi-)