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GRID APPROXIMATION OF A SINGULARLY PERTURBED PARABOLIC EQUATION WITH DEGENERATING CONVECTIVE TERM AND DISCONTINUOUS RIGHT-HAND SIDE

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This paper is dedicated to the 60th birthday of Martin Stynes

Abstract. The grid approximation of an initial-boundary value problem is considered for a singularly perturbed parabolic convection-diffusion equation with a convective flux directed from the lateral boundary inside the domain in the case when the convective flux degenerates inside the domain and the right-hand side has the first kind discontinuity on the degeneration line. The high-order derivative in the equation is multiplied by ε^2, where ε is the perturbation parameter, $\varepsilon \in (0, 1]$. For small values of ε, an *interior layer* appears in a neighbourhood of the set where the right-hand side has the discontinuity. A finite difference scheme based on the standard monotone approximation of the differential equation in the case of uniform grids converges only under the condition $N^{-1} = o(\varepsilon)$, $N_0^{-1} = o(1)$, where $N + 1$ and $N_0 + 1$ are the numbers of nodes in the space and time meshes, respectively. A finite difference scheme is constructed on a piecewise-uniform grid condensing in a neighbourhood of the interior layer. The solution of this scheme converges ε-uniformly at the rate $\mathcal{O}(N^{-1} \ln N + N_0^{-1})$. Numerical experiments confirm the theoretical results.

Key words. parabolic convection-diffusion equation, perturbation parameter, degenerating convective term, discontinuous right-hand side, interior layer, technique of derivation to *a priori* estimates, piecewise-uniform grids, finite difference scheme, ε-uniform convergence, maximum norm.

1. Introduction

At present, methods to construct special ε-uniformly convergent finite difference schemes for singularly perturbed elliptic and parabolic convection-diffusion equations are well developed for the case when the problem data are sufficiently smooth and the convective term in the equations preserves the sign (e.g., strictly positive) everywhere in the domain (see, e.g., [2, 8, 10, 13, 16] and the references therein). Special methods for singularly perturbed problems with discontinuous data and degenerating convective terms are less developed; see, e.g., the case with discontinuous data in differential equations (coefficients and the right-hand side) in [9, 13], the case with discontinuous boundary conditions in [4, 16], the case with a convective term degenerating on the domain boundary for a parabolic convection-diffusion equation in [3]. Special schemes for problems with a convective term degenerating inside the domain and discontinuous data in equations were not considered earlier.

In the present paper the grid approximation of an initial-boundary value problem is considered for a singularly perturbed parabolic convection-diffusion equation with a convective flux directed from the lateral boundary inside the domain in the

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case when the convective term, being sufficiently smooth, degenerates inside the domain and the right-hand side has the first kind discontinuity on the degeneration line. For small values of the perturbation parameter ε, an interior layer arises in a neighbourhood of the set where the right-hand side has the discontinuity; the interior layer does not arise in the case of the smooth right-hand side (see Sect. 3).

For the initial-boundary value problem, a finite difference scheme is constructed on a piecewise-uniform grid condensing in a neighbourhood of the interior layer. The solution of this scheme converges ε-uniformly in the maximum norm at the rate $\mathcal{O}(N^{-1}\ln N + N_0^{-1})$, where $N+1$ and N_0+1 are the numbers of nodes in the space and time meshes, respectively. Note that in the case of a parabolic convection-diffusion equation with sufficiently smooth problem data and positive diffusion coefficient on the definition domain, the finite difference scheme converges at the same convergence rate (see, e.g., [16] and the references therein).

Contents of the paper. The formulation of the initial-boundary value problem and the aim of the research are given in Section 2. *A priori* estimates used to construct and justify developed schemes are exposed in Section 3. Finite difference schemes on uniform and piecewise-uniform grids are studied, respectively, in Section 4 and 5. Numerical experiments to investigate the constructed schemes are shown in Section 6. Conclusions are given in Section 7.

Notation. We denote by $C^{k,k/2}$ the space of functions $u(x,\,t)$ with continuous derivatives in x up to order k and continuous derivatives in t up to order $k/2$. Henceforth, M, M_i (or m) denote sufficiently large (small) positive constants that are independent of the parameter ε and of the discretization parameters. Finally, the notation $L_{(j.k)}$ $(\overline{G}_{(j.k)}, M_{(j.k)})$ means that this operator (domain, constant) is introduced in formula $(j.k)$.

2. Problem formulation and aim of the research

2.1. On the set $\overline{G}$ with the boundary S

$$\overline{G} = G \bigcup S, \quad G = D \times (0,T], \quad D = (-d,d), \tag{2.1}$$

we consider the initial-boundary value problem for the singularly perturbed parabolic convection-diffusion equation

$$L\,u(x,t) \equiv \left\{\varepsilon^2\,\frac{\partial^2}{\partial x^2} + x\,\frac{\partial}{\partial x} - \frac{\partial}{\partial t} - 1\right\} u(x,t) = f(x,t), \quad (x,t) \in G \setminus S^{\pm}, \tag{2.2a}$$

where the function $f(x,t)$ is continuous on $\overline{G}$ for $x<0$ and $x>0$ and it has a discontinuity of the first kind on the set

$$S^{\pm} = \{x=0\} \times (0,T];$$

and thus on the set $S^{\pm}$, the following conjunction condition for the first-order derivative in x is imposed:

$$l^{\pm}u(x,t) \equiv \varepsilon\,\left[\frac{\partial}{\partial x}u(x+0,\,t) - \frac{\partial}{\partial x}u(x-0,\,t)\right] = 0, \quad (x,t) \in S^{\pm}. \tag{2.2b}$$

On the boundary S, the boundary condition is prescribed

$$u(x,t) = \varphi(x,t), \quad (x,t) \in S. \tag{2.2c}$$

As a solution of the initial-boundary value problem (2.2), (2.1) in the case of the right-hand side having the first kind discontinuity on the set $S^{\pm}$, denoted by

$$[f(x^*,t)]^j \equiv f(x^*+0,t) - f(x^*-0,t) \neq 0, \quad (x^*,t) \in S^{\pm}, \tag{2.3}$$