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SOME ERROR ESTIMATES OF FINITE VOLUME ELEMENT APPROXIMATION FOR ELLIPTIC OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper, finite volume element method is applied to solve the distributed optimal control problems governed by an elliptic equation. We use the method of variational discretization concept to approximate the problems. The optimal order error estimates in L^2 and L^∞-norm are derived for the state, costate and control variables. The optimal H^1 and $W^{1,\infty}$-norm error estimates for the state and costate variables are also obtained. Numerical experiments are presented to test the theoretical results.

Key words. finite volume element method, variational discretization, optimal control problems, elliptic equation, distributed control.

1. Introduction

The finite volume element method is a discretization technique for partial differential equations. Due to its local conservative property and other attractive properties, such as the robustness with the unstructured meshes, the finite volume element method is widely used in computational fluid dynamics. In general, two different functional spaces (one for the trial space and one for the test space) are used in the finite volume element method. Owing to the two different spaces, the numerical analysis of the finite volume element method is more difficult than that of the finite element method and finite difference method. Since the method was proposed, there have been many results in the literature. Early work for the finite volume element method can be found in [2, 5, 7, 13, 15, 19]. In [2], Bank and Rose obtain the result that the finite volume approximation is comparable with the finite element approximation in H^1-norm. The optimal L^2-error estimate is obtained in [13, 19] under the assumption that $f \in H^1$. In [19], the authors also obtain the H^1-norm and maximum-norm error estimates. In [7], Chatzipantelidis proposes a nonconforming finite volume element method and obtains the L^2-norm and H^1-norm error estimates. Recently, Ye proposes a discontinuous finite volume element method. Unified error analysis for conforming, nonconforming and discontinuous finite volume element method is presented in [16]. High order finite volume element method can be found in, e.g., [8, 14]. For other recently development, we refer readers to see [6, 18, 21, 28] and the references therein.

The optimal control problems introduced in [23] are playing an increasingly important role in science and engineering. They have various application in the

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operation of physical, social, and economic processes. Finite element method is an important numerical method for the problems of partial differential equations and widely used in the numerical solution of optimal control problems. Only for the optimal control problems governed by linear elliptic equation, there have been many results in the literature. For instance, some a priori error estimates of the finite element approximation for the optimal control problems are established in [24]. A posteriori error estimates and adaptive finite element methods are studied in [22, 24]. Some superconvergence results are reported in, e.g. [24, 25]. The error estimates of mixed finite element approximation for optimal control problems are investigated in, for example, [11, 24]. Furthermore, some superconvergence results of the mixed finite element method are obtained in, e.g., [11, 24]. Other numerical methods for optimal control problems can be seen in [3, 12, 17, 29].

In most of these papers, the state and costate (adjoint state) variables are discretized by continuous linear elements and the control variable by piecewise constant or piecewise linear polynomials. The approximate order of the control variable is $O(h)$ or $O(h^{3/2})$ in the sense of L^2-norm or L^∞-norm (see, e.g., [26]). In [20], Hinze proposes a variational discretization concept for optimal control problems with control constraints. With the variational discretization concept, the control variable is not discretized directly, but discretized by a projection (defined later, see (3.7)) of the discrete costate variable. The convergent order of the control variable is $O(h^2)$.

There are two approaches to find the approximate solution of the optimal control problems governed by partial differential equation. One is of the *optimize-then-discretize* type. One first applies the Lagrange multiplier methods to obtain an optimal system, at the continuous level, consisting of the state equation, an adjoint equation and an optimal condition. Then one use some numerical method to discretize the resulting system. The other is of the *discretize-then-optimize* type. One first discretizes the optimal control problems by some means and then applies the Lagrange multiplier rule to the resulting discrete optimization problem. The two discrete systems, determined by the two approaches, are the same when finite element method is used. But in general, these discrete systems are not the same. In [17], the streamline upwind Galerkin method is applied to approximate the solution of elliptic optimal control problems using the *optimize-then-discretize* approach. In [29], the authors also use the *optimize-then-discretize* approach to solve the optimal control problem governed by convection dominated diffusion equation.

In engineering, there exist widely optimal control problems governed by fluid flow equation. And the finite volume element method is widely used in computational fluid dynamics. To our best knowledge, there is no published result in which the finite volume element method is applied to solve the optimal control problems. We want to use finite volume element method to solve fluid optimal control problems. But here we will use the *optimize-then-discretize* approach and the *finite volume element* method to find the approximation of elliptic optimal control problems.

In this paper, we consider the following optimal control problems: Find y, u such that

$$\min_{u\in U_{ad}} \frac{1}{2}||y-y_d||^2_{L^2(\Omega)} + \frac{\alpha}{2}||u||^2_{L^2(\Omega)}, \tag{1.1}$$

$$-\nabla\cdot(A\nabla y) = Bu + f, \text{ in } \Omega, \tag{1.2}$$

$$y = 0, \text{ on } \Gamma, \tag{1.3}$$

where $\Omega \subset R^2$ is a bounded convex polygon domain and Γ is the boundary of Ω, α is a positive number, $f,\ y_d \in L^2(\Omega)$ or $H^1(\Omega)$, $A = (a_{i,j}(x))$ is a 2×2