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ERROR ANALYSIS OF LINEARIZED SEMI-IMPLICIT GALERKIN FINITE ELEMENT METHODS FOR NONLINEAR PARABOLIC EQUATIONS

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Abstract. This paper is concerned with the time-step condition of commonly-used linearized semi-implicit schemes for nonlinear parabolic PDEs with Galerkin finite element approximations. In particular, we study the time-dependent nonlinear Joule heating equations. We present optimal error estimates of the semi-implicit Euler scheme in both the L^2 norm and the H^1 norm without any time-step restriction. Theoretical analysis is based on a new splitting of error function and precise analysis of a corresponding time-discrete system. The method used in this paper is applicable for more general nonlinear parabolic systems and many other linearized (semi)-implicit time discretizations for which previous works often require certain restriction on the time-step size τ.

Key words. Nonlinear parabolic system, unconditionally optimal error estimate, linearized semi-implicit scheme, Galerkin method.

1. Introduction

In the last several decades, numerous effort has been devoted to the development of efficient numerical schemes for nonlinear parabolic PDEs arising from a variety of physical applications. A key issue to those schemes is the time-step condition. Usually, fully implicit schemes are unconditionally stable. However, at each time step, one has to solve a system of nonlinear equations. An explicit scheme is much easy in computation. But it suffers the severely restricted time-step size for convergence. A popular and widely-used approach is a linearized (semi)-implicit scheme, such as linearized semi-implicit Euler scheme. At each time step, the scheme only requires the solution of a linear system. To study the error estimate of linearized (semi)-implicit schemes, the boundedness of numerical solution (or error function) in L^∞ norm or a stronger norm is often required. If a priori estimate for numerical solution in such a norm cannot be obtained, one may employ the induction method with inverse inequality to bound the numerical solution, such as

$$\|R_h u(\cdot,t_n) - U_h^n\|_{L^\infty} \le Ch^{-d/2}\|R_h u(\cdot,t_n) - U_h^n\|_{L^2} \le Ch^{-d/2}(\tau^p + h^{r+1}), \tag{1.1}$$

where $u(\cdot,t_n)$ and U_h^n are the exact solution and numerical solution, respectively, R_h is some projection operator and d is the dimension. The above inequality, however, results in a time-step restriction, particularly for problems in three spatial dimensions. Such a technique has been widely used in the error analysis for many different nonlinear parabolic PDEs, *e.g.*, see [1, 16, 18, 20, 21] for Navier-Stokes equations, [2, 11, 36] for nonlinear Joule heating problems, [15, 25, 27] for porous media flows, [7, 12, 13, 28] for viscoelastic fluid flow, [22, 35] for KdV equations and [10, 29] for some other equations. In all these works, error estimates were established under certain time-step restrictions. We believe that these time-step restrictions may not be necessary in most cases. In this paper, we only focus our

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attention to a time-dependent and nonlinear Joule heating system by a linearized semi-implicit scheme. However, our approach is applicable for more general nonlinear parabolic PDEs and many other time discretizations to obtain optimal error estimates unconditionally.

The time-dependent nonlinear Joule heating system is defined by

$$\frac{\partial u}{\partial t} - \Delta u = \sigma(u)|\nabla\phi|^2, \tag{1.2}$$

$$-\nabla\cdot(\sigma(u)\nabla\phi) = 0, \tag{1.3}$$

for $x \in \Omega$ and $t \in [0,T]$, where Ω is a bounded smooth domain in $\mathbb{R}^d$, $d = 2, 3$. The initial and boundary conditions are given by

$$\begin{aligned} &u(x,t) = 0, \quad \phi(x,t) = g(x,t) && \text{for } x \in \partial\Omega,\ t \in [0,T], \\ &u(x,0) = u_0(x) && \text{for } x \in \Omega. \end{aligned} \tag{1.4}$$

The nonlinear system above describes the model of electric heating of a conducting body, where u is the temperature, ϕ is the electric potential, and σ is the temperature-dependent electric conductivity. Following the previous works [11, 36], we assume that $\sigma \in C^1(\mathbb{R})$ and

$$\kappa \le \sigma(s) \le K, \tag{1.5}$$

for some positive constants κ and K.

Theoretical analysis for the Joule heating system was done by several authors [3, 5, 8, 34, 31, 32, 33]. Among these works, Yuan [33] proved existence and uniqueness of a C^α solution in three-dimensional space. Based on this result, further regularity can be derived with suitable assumption on the initial and boundary conditions. Numerical methods and analysis for the Joule heating system can be found in [2, 4, 11, 30, 36, 37, 38]. For the system in two-dimensional space, optimal L^2 error estimate of a mixed finite element method with the linearized semi-implicit Euler scheme was obtained in [36] under a weak time-step condition. Error analysis for the three-dimensional model was given in [11], in which the linearized semi-implicit Euler scheme with a linear Galerkin FEM was used. An optimal L^2-error estimate was presented under the time step restriction $\tau \le O(h^{1/2})$. A more general time discretization with higher-order finite element approximations was studied in [2]. An optimal L^2-norm error estimate was given under the conditions $\tau \le O(h^{3/2p})$ and $r \ge 2$ where p is the order of the discrete scheme in time direction and r is the degree of piecewise polynomial approximations used. No optimal error estimates in H^1-norm have been obtained.

The main idea of this paper is a splitting of the numerical error into the temporal direction and the spatial direction by introducing a corresponding time-discrete parabolic system (or elliptic system). Error bounds of the Galerkin finite element methods for the time-discrete parabolic equations in certain norm is dependent only upon the spatial mesh size h and independent of the time-step size τ. If a suitable regularity of the solution of the time-discrete equations can be proved, numerical solution in the L^∞ norm (or stronger norm) is bounded unconditionally by the induction assumption

$$\|R_h U^n - U_h^n\|_{L^\infty} \le Ch^{-d/2}\|R_h U^n - U_h^n\|_{L^2} \le Ch^{-d/2}h^{r+1} \tag{1.6}$$

where U^n is the solution of the time-discrete equations. With the boundedness, optimal error estimates can be established for the fully discrete scheme without any time-step restriction. In this paper, we analyze the linearized (semi-implicit)