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ON LINEAR FINITE ELEMENTS FOR SIMULTANEOUSLY RECOVERING SOURCE LOCATION AND INTENSITY

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Abstract. Linear elements are least expensive finite elements for simultaneously recovering the source location and intensity in a general convection-diffusion process. However, the derivatives of the least-squares objective functional with Tikhonov regularizations are not well-defined when linear finite elements are used. In this work we provide a systematic formulation of the numerical inversion using linear finite elements and propose some effective techniques to overcome the undefinedness that may occur in inversion process. We show that linear finite elements can be made very robust and efficient in simultaneously recovering the source location and intensity. Numerical results are presented to validate the robustness and effectiveness of the proposed algorithm.

Key words. source location, source intensity, convection-diffusion and linear finite elements.

1. Introduction

Source isolation is an effective measure taken in practical applications for pollution prevention in groundwater, lakes and rivers, and for quick response to fire accidents or some attacks caused by airborne/aerosolized chemical or biological agent. The knowledge of the contaminant sources and their intensities plays an essential role in taking the measures. When the source location and intensity are available, one may simulate the distribution of the pollutant concentration and its transport process in the concerned environmental systems by a transport or convection-diffusion model [1, 11, 10, 15]. This is the so-called forward problem. But in many applications such as those aforementioned ones, the source location and intensity are the essential information one needs to make further actions. This can be achieved usually by solving an inverse transport or convection-diffusion problem, using some extra data of concentration or its fluxes measured by appropriately located sensors within a certain period of time. This inversion process may help recover the location, intensity or time release history of the pollutant sources. As a typical inverse problem the identification of the source location and intensity is severely ill-posed in the sense of Hadamard [5, 18, 19], that is, one of the three fundamental properties such as the existence, uniqueness and stability of solutions to the considered problem is not satisfied. For numerical processes, the most important property is the stability. The inverse problem of recovering the source location and intensity is typically unstable, i.e., small noise in the observation data may cause tremendous change of the source location and intensity.

Various approaches have been applied to the inverse problem of recovering the source intensity distribution [16, 17, 20] when the source location is known, or of recovering both the source intensity and location [1, 6, 7, 8, 11]. Quasi-explicit reconstruction formulae may exist for point source location recovery in one-dimensional cases under some careful design of the observation locations [7, 8], and the point source locations may be found by solving some integral equation involving the

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fundamental solutions in two dimensions for a heat conduction problem [9]. Regarding to the source intensity recovery, explicit formula is available for inversion in one-dimensional cases with constant coefficients and steady sources [11]. The method of quasi-reversibility was studied in [17] to retrace the plume history for one-dimensional cases, and a fourth-order term was added to the model to stabilize the numerical dispersion and counteract the data noise.

Optimization approaches are popular in solving various inverse problems [3, 13, 14, 21]. Linear programming and multiple regression method were applied in [6] for finding the pollutant source location and their magnitude in a simple one-dimensional model. When the source location is known, some numerical efforts were made to recover the time intensity of the source, e.g., in [16] for one dimension and in [20] for two dimensions. The Tikhonov regularization approach was used in [16], while an iterative method was applied in [20] without any regularization, so it works only for noise-free observation data, and performs poorly whenever there is noise present in the data.

In this work we will apply the popular output least-squares formulation with appropriate regularizations for simultaneously recovering the point source location and the corresponding time-dependent intensity profile. Our major investigation will be on the space discretization of the resulting nonlinear optimization system by the least expensive finite element method, namely the piecewise linear finite elements. There are two major reasons why we are interested in the use of linear elements. First, higher order finite elements are much more expensive than linear elements, especially in three-dimensions. Second, unlike for direct problems, higher order continuous finite elements do not bear any advantages over the linear elements for solving inverse problems. This is due to the fact that the solution of an inverse problem usually requires solving an adjoint problem corresponding to the forward model equation, but the regularity of the solution to the adjoint problem is usually low, often lower than H^2, no matter whether the domains or the coefficients involved in the concerned equations are smooth or not since the measurement data always serve as a source term of the adjoint problem and the data always contain noise in applications. Therefore high-order elements can not generate better accuracy than linear elements for inverse problems. The above two reasons show that linear elements are more practical and reasonable in terms of both accuracy and computational efficiency. This motivates our current investigation of linear elements for the inverse problem of simultaneously recovering the source location and its corresponding intensity. To the best of our knowledge, this practically important topic has not been studied in the literature. A major difficulty arises from the space singular delta function in the source term of the convection-diffusion model and the time singular delta function in the adjoint system. As we shall see, for the reconstruction of the source location we need to evaluate the derivatives of functions from the finite element space used, but that are unfortunately not well-defined along element edges (2D) or faces (3D) for piecewise linear finite elements. This difficulty can be naturally avoided by using higher order C^1 finite elements [4], but with much higher computational efforts, since many more degrees of freedom will be involved than the ones for linear elements. We shall propose two techniques to overcome such difficulty and show that linear finite elements can be made very robust and effective in simultaneously recovering the source location and intensity. So this work provides a complete formulation of the numerical inversion process using the linear finite elements.