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UNIFIED A POSTERIORI ERROR ESTIMATOR FOR FINITE ELEMENT METHODS FOR THE STOKES EQUATIONS

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Abstract. This paper is concerned with residual type a posteriori error estimators for finite element methods for the Stokes equations. In particular, the authors established a unified approach for deriving and analyzing a posteriori error estimators for velocity-pressure based finite element formulations for the Stokes equations. A general a posteriori error estimator was presented with a unified mathematical analysis for the general finite element formulation that covers conforming, non-conforming, and discontinuous Galerkin methods as examples. The key behind the mathematical analysis is the use of a lifting operator from discontinuous finite element spaces to continuous ones for which all the terms involving jumps at interior edges disappear.

Key words. A posteriori error estimate, finite element methods, Stokes equations

1. Introduction

A posteriori error estimator refers to a computable formula that offers a measure for judging the reliability and efficiency of a particular numerical scheme employed for approximating the solution of partial differential equations or alike. With a mathematically justified a posteriori error estimator, one would be able to generate a mesh that is tailored at reducing computational errors at places of great need. This process is commonly known as *adaptive mesh refinement* which has become a useful and important tool in today's scientific and engineering computing. The goal of this paper is to offer a systematic framework for developing and analyzing a posteriori error estimators for finite element methods for model Stokes equations.

This paper is concerned with residual type a posteriori error estimators. In other words, the computable formula for judging the efficiency and reliability of numerical schemes shall be given by functions of residuals. Along this avenue, several fine results have been developed for finite element methods for the Stokes equations. For conforming finite element methods, some a posteriori error estimators have been derived for mini-elements by Verfurth [21] and Bank-Welfert [4]. Ainsworth-Oden [3] and Nobile [18] have considered more general conforming finite elements in their study. For nonconforming finite elements, a posteriori error estimation for the Crouzeix-Raviart element [8] has been developed by several researchers such as Verfurth [22], Dari-Durán-Padra [9] and Doerfler-Ainsworth [10]. Carstensen, Gudi, and Jensen [5] proposed and analyzed an a posteriori error estimator for discontinuous Galerkin methods by using a stress-velocity-pressure formulation for the Stokes equations. Kay and Silvester [16] established a posteriori error estimation

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for the stabilized finite element formulation. The recovery based a posteriori error estimate for the Stokes equations is investigated in [12].

In both [9] and [10], the analysis for their a posteriori error estimators was based on a Helmholtz decomposition for decomposing the Crouzeix-Raviart element into two parts: an exactly divergence-free part and the second as its orthogonal complement. While the Helmholtz decomposition offers an applaudable approach for analyzing the efficiency and reliability of a posteriori error estimators for the Stokes equations, the method has difficulty in being extended to finite element approximations arising from discontinuous Galerkin methods. The main difficulty comes from the fact that the approximate velocity field from the discontinuous finite element methods is not divergence-free in the classical sense. Therefore, other analytical techniques have been developed for discontinuous finite elements; but most of them requires special and unnecessary properties about the finite element mesh. For example, Houston, Schötzau and Wihler [14] have developed an a posteriori error analysis for the discontinuous $Q_k - Q_{k-1}$ element on partitions consisting of parallelograms only.

In this paper, we establish a unified approach for deriving and analyzing a posteriori error estimators of residual type for velocity-pressure based formulations of the Stokes equations. In particular, we shall develop a general finite element formulation that covers conforming, non-conforming, and discontinuous Galerkin methods as examples. Then, a general a posteriori error estimator shall be presented with a unified mathematical analysis. The key behind the analysis is the use of a lifting operator from discontinuous finite element spaces to continuous ones for which all the terms involving jumps at interior edges disappear. A similar lifting operator was employed by Karakashian and Pascal [15] for analyzing a posteriori error estimates for a discontinuous Galerkin approximation to second order elliptic equations.

The paper is organized as follows. In Section 2, a model Stokes problem and some notations are introduced. In Section 3, we shall first present a general finite element formulation for the Stokes equations, and then illustrate how most existing conforming, nonconforming, and discontinuous Galerkin methods be represented by the general framework. In Section 4, we establish an analytical tool for analyzing the general a posteriori error estimator of residual type. Finally in Section 5, we present some numerical results to confirm the theory developed in previous sections.

2. Preliminaries and notations

Let Ω be an open bounded domain in $\mathbb{R}^d, d = 2, 3$. Denote by $\partial\Omega$ the boundary of Ω. The model problem seeks a velocity function $\mathbf{u}$ and a pressure function p satisfying

$$-\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \tag{2}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \tag{3}$$

where Δ, ∇, and $\nabla\cdot$ denote the Laplacian, gradient, and divergence operators, respectively, and $\mathbf{f}$ is the external volumetric force acting on the fluid.

For simplicity, the algorithm and its analysis will be presented for the model Stokes problem (1)-(3) only in two-dimensional spaces (i.e.; $d = 2$) with polygonal domains. An extension to the Stokes problem in three dimensions can be made formally for general polyhedral domains.