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LOCAL ERROR ESTIMATES OF THE LDG METHOD FOR 1-D SINGULARLY PERTURBED PROBLEMS

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Abstract. In this paper local discontinuous Galerkin method (LDG) was analyzed for solving 1-D convection-diffusion equations with a boundary layer near the outflow boundary. Local error estimates are established on quasi-uniform meshes with maximum mesh size h. On a subdomain with $O(h\ln(1/h))$ distance away from the outflow boundary, the L^2 error of the approximations to the solution and its derivative converges at the optimal rate $O(h^{k+1})$ when polynomials of degree at most k are used. Numerical experiments illustrate that the rate of convergence is uniformly valid and sharp. The numerical comparison of the LDG method and the streamline-diffusion finite element method are also presented.

Key words. Local discontinuous Galerkin method, singularly perturbed, local error estimates.

1. Introduction

We are interested in the convection-diffusion problem

$$\begin{aligned} -\epsilon u'' + au' + bu &= f \qquad \text{in } \mathcal{I} = (0,1), \\ u &= 0 \qquad \text{on } \partial\mathcal{I} = \{0,1\}, \end{aligned} \tag{1.1}$$

where $0 < \epsilon \ll 1$ is the diffusion parameter, $a = a(x) \geq \alpha > 0$ accounts for the convection, and $b = b(x)$ accounts for the reaction term. The function $f = f(x)$ is a given source term. We assume that α is a constant; a, b, and f are sufficiently smooth on $\overline{\mathcal{I}}$.

When ϵ is small, the solution to Problem (1.1) typically has a boundary layer with width $O(\epsilon \ln \frac{1}{\epsilon})$ at $x = 1$. The standard finite element method produces numerical solutions that exhibits nonphysical oscillation on uniform mesh unless the mesh size is comparable with ϵ. Many techniques have been developed to eliminate the nonphysical oscillation (c.f. [1, 11, 12, 15, 16]). Among these techniques is the streamline-diffusion finite element method (SDFEM) proposed in eighties by Hughes et.al. (c.f. [12]) by adding an appropriate amount of artificial diffusion in the streamline direction to stabilize the conforming finite element method. The SDFEM is quite satisfactory for practical situations, but may lead to large artificial layers near boundaries and discontinuities. There has been many theoretical results published up to now (c.f. [6, 14, 16]). Another technique is to employ a layer-adapted mesh based on the a priori knowledge of Problem (1.1), such as Shishkin-type meshes, Bakhvalov-type meshes (c.f. [15, 16, 20, 21]).

Starting from 1970's, discontinuous Galerkin methods has been intensively studied and applied to hyperbolic and convection-dominated elliptic problems with great success (c.f. [7, 8, 9, 13]). Recently, the superconvergence of the numerical traces and the L^2 convergence of DG methods have been discussed for one-dimensional convection-diffusion problems (c.f. [4, 5, 18, 19, 21]). It has been reported in the

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numerical experiments of [18] that the error curves of numerical traces didn't show any any oscillation even on uniform meshes if mesh size h is comparable with ϵ, or if $\epsilon \ll h$ is extremely small. It implies that the local discontinuous Galerkin method (LDG) seems not to produce a large artificial layers as SDFEM did outside the boundary layer region of Problem (1.1). Motivated by this finding, we are interested in investigating the LDG method for Problem (1.1) on uniform or quasi-uniform meshes to see how efficient it could be.

In this work, we proved that the L^2 errors of $u' - \epsilon^{-1}Q$ and $u - U$ converge at the optimal rate $O(h^{k+1})$ on a subdomain $\mathcal{I}_0 \subset \mathcal{I}$ where $\partial\mathcal{I}_0$ is $O(h\ln(1/h))$ distance away from the outflow boundary of $\mathcal{I}$, i.e., $x = 1$. Here (U, Q) denotes the LDG approximation of $(u, \epsilon u')$; h denotes the maximum mesh size; and approximation space consists of piecewise polynomials of degree at most k. These rates of convergence are uniformly valid in terms of the singular perturbation parameter ϵ, as verified by our numerical experiments. The numerical comparison of LDG and SDFEM are also presented in this paper. The numerical results in Section 4 illustrate that the L^2 errors of the LDG approximations to the exact solution and its derivative on $\mathcal{I}_0$ are smaller comparing with the L^2 errors of SDFEM on the same subdomain $\mathcal{I}_0$. For a fixed uniform mesh, the subdomain $\mathcal{I}_0$ of the LDG method expands and contains more mesh elements as the parameter $\epsilon \to 0$. If $\epsilon \ll h$ is extremely small, the error curves of numerical traces will not show any oscillation. Furthermore, numerical results shows that a small artificial layer does exist for small ϵ if the mesh size h is not very large.

On the other hand, the subdomain $\mathcal{I}_0$ of SDFEM expands slower than LDG and the artificial layer always contains $(k+1)\ln N$ mesh elements as $\epsilon \to 0$. Therefore, its nodal error curves will always show an oscillation near the outflow boundary $x = 1$ even if ϵ is extremely small. This finding, then, seems to support the former view in [18] that the DG method is more 'local' than finite element method.

The outline of this article is as follows: In Section 2, we present the LDG discretization and state our main results, which give some local error estimates. The proof of the main results is carried out in details in Section 3. In section 4, we present several numerical experiments testing our theoretical results. We end in Section 5 with some concluding remarks.

Notations. Throughout this article, the letter C will denote a generic constant not necessarily the same at each occurrence. It might depend on the coefficient functions a, b, the right-hand side function f, and the polynomial degree k, but is independent of the singular perturbation parameter ϵ and the mesh. For any measurable subdomain $D \subseteq \mathcal{I}$, we use the standard Sobolev spaces $L^2(D)$, $H^1(D)$, $H^s(D) = W_2^s(D)$ for some nonnegative integer s.

2. The LDG discretization and main results

In this section, we present the LDG discretization and state our main results. We begin with partitioning the domain $\mathcal{I}$. If $0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = 1$, we denote by $\mathcal{I}_h = \{I_j = (x_{j-1}, x_j), j = 1, 2, \cdots, N\}$ a quasi-uniform partition of domain $\mathcal{I}$, and by $h_j = x_j - x_{j-1}$ the length of the j-th element. Let $h = \max\limits_{j=1,\cdots,N} h_j$. For any $j = 1, 2, \cdots, N$, there exists a constant C_q such that $h_j \geq C_q h$. Define $v(x_j^{\pm}) = \lim_{\delta\to 0} v(x_j \pm \delta)$ as in [13]. For each element $I_j \in \mathcal{I}_h$, we set its outward unit normal $n_{I_j}(x_j) = 1$ and $n_{I_j}(x_{j-1}) = -1$. We denote $v_j = v(x_j)$, $v_j^{\pm} = v(x_j^{\pm})$, $[\![v_0]\!] = -v_0^+$ and $[\![v_N]\!] = v_N^-$, $[\![v_j]\!] = v_j^- n_{I_j}(x_j) + v_j^+ n_{I_{j+1}}(x_j) = v_j^- - v_j^+$ for $j = 1, \cdots, N-1$.