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THE HALF-PLANES PROBLEM FOR THE LEVEL SET EQUATION

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Abstract. The paper is dedicated to the construction of an analytic solution for the level set equation in $\mathbb{R}^2$ with an initial condition constituted by two half-planes. Such a problem can be seen as an equivalent Riemann problem in the Hamilton-Jacobi equation context. We first rewrite the level set equation as a non-strictly hyperbolic problem and obtain a Riemann problem where the line sharing the initial discontinuity corresponds to the half-planes junction. Three different solutions corresponding to a shock, a rarefaction and a contact discontinuity are given in function of the two half-planes configuration and we derive the solution for the level set equation. The study provides theoretical examples to test the numerical methods approaching the solution of viscosity of the level set equation. We perform simulations to check the three situations using a classical numerical method on a structured grid.

Key Words. Riemann problem, Cauchy problem, level set equation, analytical solution, half-planes problem

1. Introduction

The interface tracking problem takes place in various fields like front flame propagation, crystal growth in solidification process, fluid-structure interaction with moving solid boundary, computer vision, dynamics bubbles or drops for example. The level set method (see [8] for an overview) consists in representing the free boundary as the zero-level of a continuous function ϕ where the normal velocity F is a prescribed function.
Consider the Cauchy problem:

$$\begin{cases} \partial_t \phi(\boldsymbol{x},t) + F\,|\nabla\phi(\boldsymbol{x},t)| = 0 & \text{in } \mathbb{R}^2 \times [0,T], \\ \phi(\boldsymbol{x}, t=0) = \phi_0(\boldsymbol{x}) & \text{in } \mathbb{R}^2, \end{cases} \tag{1}$$

where $F(\boldsymbol{x},t)$ is a given Lipschitz function on $\mathbb{R}^2 \times [0,T]$ while ϕ_0 is a Lipschitz function on $\mathbb{R}^2$. Existence and uniqueness of the Lipschitz viscosity solution for problem (1) on $\mathbb{R}^2 \times [0,T]$ is proved (see [1, 7]).

Since ϕ is a Lipschitz function, vector $\mathbf{U} = \nabla\phi$ is a bounded vector-valued function and applying the gradient operator to equation (1), we derive the Cauchy problem for the conservation law system associated to the level set equation:

$$\begin{cases} \partial_t \mathbf{U}(\boldsymbol{x},t) + \nabla(F\,|\mathbf{U}(\boldsymbol{x},t)|) = 0 & \text{in } \mathbb{R}^2 \times [0,T], \\ \mathbf{U}(\boldsymbol{x}, t=0) = \nabla\phi_0(\boldsymbol{x}) & \text{in } \mathbb{R}^2. \end{cases} \tag{2}$$

For the one dimensional situation, [7] proves that the Lipschitz viscosity solution ϕ of equation (1) corresponds to the bounded entropic solution U of equation (2) with $U = \partial_x\phi$. Such a result is not established for higher dimension since we do not have the uniqueness of solution for the non-strictly hyperbolic system (2) [2]. Nevertheless, if we regularize equations (1) and (2) adding a diffusion term $\varepsilon\Delta\phi$ and $\varepsilon\Delta U$ respectively, the new solutions satisfy $\mathbf{U}_\varepsilon = \nabla\phi_\varepsilon$. Passing to the limit assuming a L^∞ convergence of ϕ_ε

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toward ϕ and a L^1 convergence of U_ε toward U, we have that $\mathbf{U} = \nabla\phi$ where ϕ is the viscosity solution and U the entropy solution.

From a numerical point of view, the non-strictly hyperbolic system discretization using the finite volume method leads to solve Riemann problems for each cell interface. The problem is reduced to a one dimensional hyperbolic equation with a discontinuous flux function [3] making difficult a complete and explicit resolution. We propose a different approach based on the following remark: a constant state for the non-strictly hyperbolic problem (2) corresponds to a plane for the level set equation (1). Hence we propose to study an equivalent Riemann problem in the level set equation context using two half-planes as an initial continuous condition [4, 5].

2. The two half-planes problem

For the sake of simplicity, we assume in the following that function F is reduced to a constant function. Such an assumption is not restrictive since we usually solve Riemann problems using the normal velocity evaluated at the interface midpoint.
Let π_1 and π_2 be two planes of the $(\boldsymbol{x}, z) = (x_1, x_2, z) \in \mathbb{R}^3$ space. We impose that the planes contain the origin point and the plane equations write

$$z = \phi_i(\boldsymbol{x}) = \mathbf{U}_i.\boldsymbol{x}, \quad i = 1, 2$$

where $\mathbf{U}_i$ are given vectors of $\mathbb{R}^2$. Of course, when the two planes are equal, one has $\phi_0(\boldsymbol{x}) = \phi_1(\boldsymbol{x}) = \phi_2(\boldsymbol{x})$ as an initial condition and the solution is given by $\phi(\boldsymbol{x}, t) = \phi_0(\boldsymbol{x}) - F|\mathbf{U}_L|t$ with $\mathbf{U}_L = \mathbf{U}_1 = \mathbf{U}_2$ which corresponds to a simple translation of the initial plane with velocity $F|\boldsymbol{U}_L|$ (see figure (1)).

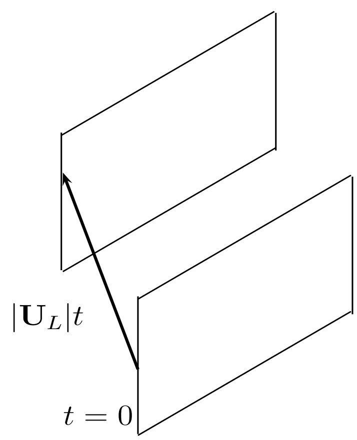

Figure 1: *The trivial case when* $\phi_0 = \phi_1 = \phi_2$*. The solution representation in* $\mathbb{R}^3$ *corresponds to the initial plane translated following the* Oz *axis with velocity* $F|\boldsymbol{U}_L| = F|\nabla\phi_0|$ *(we take* $F = 1$ *in the figure).*

Now, we consider the nontrivial case when the two planes are different. To construct an initial condition ϕ_0 for the Cauchy problem (1) we consider an arbitrary line $\delta \subset \mathbb{R}^2$ passing to the origin, an arbitrary normal vector $\mathbf{W} \in \mathbb{R}^2$ and we define the left part $\mathcal{P}_L$ and the right part $\mathcal{P}_R$ of $\mathbb{R}^2$ such that $\mathbf{W}$ goes from left to right (see figure (2)). We then construct the initial condition by $\phi_0(\boldsymbol{x}) = \phi_1(\boldsymbol{x})$ if $\boldsymbol{x} \in \mathcal{P}_L$ and $\phi_0(\boldsymbol{x}) = \phi_2(\boldsymbol{x})$ if $\boldsymbol{x} \in \mathcal{P}_R$. Such a definition gives rise to a function which is not *a priori* continuous on δ and disqualify the construction since we would like to handle continuous solutions. It results that the interface δ can not be arbitrary but has to be defined such that we have a