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EXTRAPOLATION OF THE FINITE ELEMENT METHOD ON GENERAL MESHES

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Abstract. In this paper, we consider the extrapolation method for second order elliptic problems on general meshes and derive a type of finite element expansion which is dependent of the triangulation. It allows to prove the effectiveness of the extrapolation on general meshes and also validates the extrapolation method can be applied on the automatically produced meshes of the general computing domains. Some numerical examples are given to illustrate the theoretical analysis.

Key words. Extrapolation, finite element method, general meshes.

1. Introduction

It is well known that the extrapolation method, which was established by Richardson in 1926, is an efficient procedure for increasing the solution accuracy of many problems in numerical analysis. The effectiveness of this technique relies heavily on the existence of an asymptotic expansion for the error. The application of this approach in finite difference method can be found in the book of Marchuk and Shaidurov [11]. This technique has been well demonstrated in the frame of the finite element method [7, 10, 9, 5].

Usually in the finite element method, we first need to get the error expansion for the solution approximations such as [7, 2, 10, 9, 5]

$$u_h(x) - \pi_h u(x) = c_1(u)h^k + O(h^{k+\delta}), \tag{1}$$

in some norm sense, where c_1 is a function depending on u and independent of h, $\delta > 0$, u_h and $\pi_h u$ are the finite element approximation and interpolation, respectively. Then we can use the extrapolation method ([7, 2, 10, 9])

$$u_h^{\text{extra}} := \frac{2^k u_{h/2} - u_h}{2^k - 1}, \tag{2}$$

which has higher convergence order $O(h^{k+\delta})$ only at the mesh nodes ([7]).

If we want to obtain globally higher order convergence, we must need to apply the higher order interpolation postprocessing operator $\mathcal{Q}_h$ ([7, 9, 5])

$$u_h^{\text{extra}} := \frac{2^k \mathcal{Q}_{h/2} u_{h/2} - \mathcal{Q}_h u_h}{2^k - 1}, \tag{3}$$

which has globally higher convergence order $O(h^{k+\delta})$.

So far there are two types of extrapolation schemes for the finite element method as described above: mesh nodes extrapolation and extrapolation based on the interpolation postprocessing. So, the key for the extrapolation of the finite element method is whether we can get the expansion (1) for the finite element approximation. But, so far the expansion (1) almost need structured meshes ([7, 2, 8, 10, 9, 5]).

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So far, we always study the extrapolation situation under the structured mesh and the mesh condition is the important restrict for the extrapolation method extended to general meshes. In this paper, we first consider the interpolation expansion on general meshes and then derive what kind of needed properties of the meshes to improve the accuracy of the finite element approximations by extrapolation method. For this aim, we derive the definition of the mesh measurement for the finite element extrapolation. And based on the properties of the mesh measurement, we can obtain that the extrapolation method always has effectiveness on general meshes.

For simplicity, we consider the following second order elliptic problem

$$B(u,v) \;=\; \int_\Omega (\mathcal{A}\nabla u\cdot\nabla v+\rho uv)dxdy = f(v),\ \forall v\in\mathcal{V}:=H_0^1(\Omega), \tag{4}$$

where $\mathcal{A}=\{a_{ij}\}_{1\le i,j\le 2}\in\mathcal{R}^{2\times 2}$ is a symmetric positive definite matrix, $\rho\ge 0$ in Ω, $f(\cdot)$ a bounded linear functional in $H^{-1}(\Omega)$, and Ω is a bounded domain in $\mathcal{R}^2$ with Lipschitz boundary $\partial\Omega$. For simplicity, we assume the matrix $\mathcal{A}$ and function ρ are smooth enough.

Let $\mathcal{T}_h$ be the consistent triangulation of the domain Ω in the set of triangular elements and satisfy the following quasi-uniform condition:

$$\exists\sigma>0 \text{ such that }\ h_K/\tau_K>\sigma,\ \ \forall K\in\mathcal{T}_h$$

and

$$\exists\gamma>0,\ \text{such that}\ \ \max\{h/h_K,\ K\in\mathcal{T}_h\}\le\gamma,$$

where h_K is the diameter of K; τ_K is maximum diameter of the inscribed circle in $K\in\mathcal{T}_h$; and $h:=\max\{h_K, K\in\mathcal{T}_h\}$.

The linear finite element space $\mathcal{V}_h$ on $\mathcal{T}_h$ is defined as follows:

$$\mathcal{V}_h=\big\{v\in H^1(\Omega), v|_K\in\mathcal{P}_1(K),\ \forall K\in\mathcal{T}_h\big\}\cap H_0^1(\Omega),$$

where $\mathcal{P}_1=\text{span}\{1,x,y\}$. For our analysis, we need to define the interpolation operator $\pi_h: H^2(\Omega)\mapsto\mathcal{V}_h$ on the mesh $\mathcal{T}_h$ as

$$\pi_h u(Z_i)=u(Z_i),\quad i=1,2,3,$$

where Z_i are the three vertices of element $K\in\mathcal{T}_h$.

Based on the finite element space $\mathcal{V}_h$, we define the Ritz-projection operator $\mathcal{L}_h:\mathcal{V}\mapsto\mathcal{V}_h$ as

$$B(\mathcal{L}_h u, v_h)=f(v_h),\quad \forall v_h\in\mathcal{V}_h. \tag{5}$$

It is known about the convergence rate that

$$\|\mathcal{L}_h u-u\|_0+h\|\mathcal{L}_h u-u\|_1\le Ch^2\|u\|_2, \tag{6}$$

where $\|\cdot\|_0$ denotes the L^2-norm.

In order to use the extrapolation method, we need to refine the mesh $\mathcal{T}_h$ in the regular way. Each element $K\in\mathcal{T}_h$ is subdivided into 4 congruent triangles by connecting the midpoints of its edges (see Figure 3) and we get the finer mesh $\mathcal{T}_{h/2}$. In the similar way, we can define the finite element space $\mathcal{V}_{h/2}$ and the corresponding operators $\pi_{h/2}$, $\mathcal{L}_{h/2}$ on the finer mesh $\mathcal{T}_{h/2}$. It is obviously $\mathcal{V}_h\subset\mathcal{V}_{h/2}$.

Other notations for Sobolev spaces and norms in them (including with fractional orders) are standard and can be found in many sources like [4].

The rest of the paper is organized in the following way. In section 2 we give some useful preliminary lemmas. Interpolation expansions are obtained in section 3. Section 4 is devoted to deriving the asymptotic error expansion of the finite element approximation. The extrapolation method is discussed in Section 5. In