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A PHASE FIELD BASED PDE CONSTRAINED OPTIMIZATION APPROACH TO TIME DISCRETE WILLMORE FLOW

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Abstract. A novel phase field model for Willmore flow is proposed based on a nested variational time discretization. Thereby, the mean curvature in the Willmore functional is replaced by an approximate speed of mean curvature motion, which is computed via a fully implicit variational model for time discrete mean curvature motion. The time discretization of Willmore flow is then performed in a nested fashion: in an outer variational approach a natural time discretization is setup for the actual Willmore flow, whereas for the involved mean curvature the above variational approximation is taken into account. Hence, in each time step a PDE-constrained optimization problem has to be solved in which the actual surface geometry as well as the geometry resulting from the implicit curvature motion time step are represented by phase field functions. The convergence behavior is experimentally validated and compared with rigorously proved convergence estimates for a simple linear model problem. Computational results in 2D and 3D underline the robustness of the new discretization, in particular for large time steps and in comparison with a semi-implicit convexity splitting scheme. Furthermore, the new model is applied as a minimization method for elastic functionals in image restoration.

Key Words. phase field approach, Willmore flow, image restoration, PDE-constrained optimization.

1. Introduction

In this paper a new phase field model for the time discretization of Willmore flow, also known as elastic flow, is proposed. Willmore flow is the L^2-gradient flow for the Willmore energy

$$w[x] = \frac{1}{2}\int_{\Gamma[x]} \mathbf{h}^2 \mathrm{d}\mathcal{H}^{d-1} \tag{1}$$

on hypersurfaces $\Gamma[x] \subset \mathbb{R}^d$ parametrized over itself by the identity mapping x, where $\mathbf{h}$ is the mean curvature of $\Gamma[x]$ and $\mathcal{H}^{d-1}$ represents the $(d-1)$-dimensional Hausdorff measure. Physically, this energy reflects an approximation of the stored energy in a thin elastic shell. Applications of the Willmore energy and Willmore flow range from the modeling of edge sets in imaging [40, 39, 57, 8] to applications in surface modeling [54, 5, 4, 47, 56]. An extension of the Willmore energy, the Helfrich model, is used to describe elastic cell membranes in biology [30, 51, 21].

Willmore surfaces, defined as minimizers of the Willmore energy [55], and Willmore flow have attracted a lot of attention over the last decade. Simonett proved

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in [52] the existence of a unique and locally smooth solution of Willmore flow for sufficiently smooth initial surfaces. Furthermore, he proved exponential convergence to a sphere for initial surfaces close to a sphere. The analytic treatment of Willmore flow of curves and surfaces was investigated by Polden [45, 46] already in 1996. Kuwert and Schätzle treated long time existence and regularity of solutions in [31, 32, 33]. Recently, Rivière [48] extended results of Kuwert and Schätzle [34] for co-dimension 1 to arbitrary co-dimension.

A theoretical and numerical treatment of Willmore flow of curves was presented by Dziuk, Kuwert and Schätzle in [24]. Concerning the numerical approximation of parametric Willmore flow of surfaces Rusu [50] proposed a mixed method for the surface parametrization x and the mean curvature vector $\mathbf{h}\, n$ (with n being the surface normal) as independent variables, see also [11] for the application to surface restoration. A level set formulation was given in [20] based on a different type of splitting, involving the level set function ϕ and a curvature density function $\mathbf{h}\,|\nabla \phi|$. An error analysis for spatially discretized, time-continuous Willmore flow for graphs was presented by Deckelnick and Dziuk in [15]. They used an analogous splitting in the context of piecewise linear finite elements and proved $L^\infty(L^2)$- as well as $L^2(L^2)$-error bounds of $O(h^2 \log h)$ for the discretized graph solution. Deckelnick and Schieweck demonstrated convergence of a conforming finite element approximation in case of axially symmetric surfaces [17]. An error analysis in the case of the elastic flow of curves was recently presented by Dziuk and Deckelnick in [18]. Barrett, Garcke and Nürnberg [2] and Dziuk [25] presented alternative finite element algorithms for parametric Willmore flow. The Willmore functional is invariant with respect to Möbius transformations. In [6] Bobenko and Schröder proposed a discrete Willmore flow scheme which takes into account a circle pattern on the surface, whose temporal evolution directly reflects these invariances.

In this paper we discuss Willmore flow in the context of a phase field model. In their pioneering paper [38] Modica and Mortola proved the Γ-convergence of $a^\varepsilon[u] = \frac{1}{2}\int_\Omega \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}\Psi(u)\,\mathrm{d}x$ to the area functional, where Ψ is a properly chosen double well function. This motivated the use of a corresponding phase field model for the mean curvature motion as the L^2-gradient flow for the area functional [37]. Nochetto, Paolini and Verdi treated in [42, 41] the error between the exact evolution of an interface under mean curvature flow and the evolution of a diffusive interface computed via a phase field mean curvature motion model. They proved an optimal error estimate of order $O(\varepsilon^2)$. More recently, Evans, Soner and Souganidis proved in [28] that a scaled Allen–Cahn equation leads to a generalized motion by mean curvature. De Giorgi conjectured that the functional $w^\varepsilon[u] = \frac{1}{2\varepsilon}\int_\Omega \left(-\varepsilon\Delta u + \frac{1}{2\varepsilon}\Psi'(u)\right)^2 \mathrm{d}x$, whose integrand is the squared first variation of $a^\varepsilon[u]$, Γ-converges to the Willmore functional [14]. This functional has been investigated analytically by Loreti and March in [35] and Bellettini and Mugnai in [3]. Du et al. proved in [23] by formal asymptotics that the Euler–Lagrange equation of the phase field formulation converges to the Euler–Lagrange equation of the Willmore energy (1). For a modified functional a corresponding Γ-convergence result could finally be established by Röger and Schätzle [49]. Dondl, Mugnai and Röger used a phase field model for minimizing Euler's elastica energy of non-overlapping curves in a bounded domain [19]. Concerning numerically discretized phase field models, Chen et al. proved in [9] that the zero level set of the solution of the Allen–Cahn equation converges to the mean curvature flow as ε goes to zero if h, $\sqrt{\tau} = O(\varepsilon^p)$