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A CONFORMING DISCONTINUOUS GALERKIN FINITE ELEMENT METHOD: PART II

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Abstract. A conforming discontinuous Galerkin (DG) finite element method has been introduced in [19] on simplicial meshes, which has the flexibility of using discontinuous approximation and the simplicity in formulation of the classic continuous finite element method. The goal of this paper is to extend the conforming DG finite element method in [19] so that it can work on general polytopal meshes by designing weak gradient ∇_w appropriately. Two different conforming DG formulations on polytopal meshes are introduced which handle boundary conditions differently. Error estimates of optimal order are established for the corresponding conforming DG approximation in both a discrete H^1 norm and the L^2 norm. Numerical results are presented to confirm the theory.

Key words. Weak Galerkin, discontinuous Galerkin, stabilizer/penalty free, finite element methods, second order elliptic problem.

1. Introduction

We consider Poisson equation with a homogeneous Dirichlet boundary condition in d dimension as our model problem for the sake of clear presentation. This conforming DG method can also be used to solve other elliptic problems. The Poisson problem seeks an unknown function u satisfying

$$-\Delta u = f \quad \text{in } \Omega, \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where Ω is a bounded polytopal domain in $\mathbb{R}^d$.

The weak form of the problem (1)-(2) is given as follows: find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega). \tag{3}$$

The H^1 conforming finite element method for the problem (1)-(2) keeps the same simple form as in (3): find $u_h \in V_h \subset H_0^1(\Omega)$ such that

$$(\nabla u_h, \nabla v) = (f, v) \quad \forall v \in V_h, \tag{4}$$

where V_h is a finite dimensional subspace of $H_0^1(\Omega)$. The functions in V_h are required to be continuous that makes the classic conforming finite element formulation (4) less flexible in element construction and in mesh generation. These limitations are caused by strong continuity requirement of functions in finite element spaces. A solution to avoid these limitations is using discontinuous functions in finite element spaces.

Researchers started to use discontinuous approximation in finite element procedure in the early 1970s [2, 3, 6, 14, 18]. Local discontinuous Galerkin methods were introduced in [5]. Then a paper [1] in 2002 provides a unified analysis of discontinuous Galerkin finite element methods for Poisson equation. Since then, many new finite element methods with discontinuous approximations have been developed such as hybridizable discontinuous Galerkin method [4], mimetic finite

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differences method [7], hybrid high-order method [13], weak Galerkin method [15] and references therein.

One obvious disadvantage of discontinuous finite element methods is their rather complex formulations which are often necessary to ensure connections of discontinuous solutions across element boundaries. The purpose of this paper is to obtain a finite element formulation close to its original PDE weak form (3) for discontinuous polynomials. We believe that finite element formulations for discontinuous approximations can be as simple as follows:

$$(\nabla_w u_h, \nabla_w v) = (f, v) \quad \forall v \in V_h, \tag{5}$$

if ∇_w, an approximation of gradient, is appropriately defined for discontinuous polynomials in V_h. The formulation (5) can be viewed as a counterpart of (3) for discontinuous approximations.

In [19], we have developed a discontinuous finite element method that has an ultra simple weak formulation (5) on triangular/tetrahedal meshes for any polynomial degree $k \geq 1$. The formulation (5) has also been achieved for a WG method defined in [15] on triangular/tetrahedral meshes. The lowest order WG method developed in [15] has been improved in [8] for convex polygonal meshes, in which non-polynomial functions are used for computing weak gradient.

The purpose of this paper is to extend the conforming DG in [19] so that it can work on general polytopal meshes. The idea is to raise the degree of polynomials used to compute weak gradient ∇_w. Using higher degree polynomials in computation of weak gradient will not change the size, neither the global sparsity of the stiffness matrix. On the other side, the simple formulation of conforming DG (5) will reduce programming complexity significantly. In this paper, two conforming DG formulations on polytopal mesh are introduced for the equations (1)-(2). These two methods are different in handling the homogeneous boundary condition. Optimal order error estimates are established for the corresponding conforming DG approximations in both a discrete H^1 norm and the L^2 norm. Numerical results are presented verifying the theorem.

2. Finite Element Method

In this section, we will introduce the conforming DG method. For any given polygon $D \subseteq \Omega$, we use the standard definition of Sobolev spaces $H^s(D)$ with $s \geq 0$. The associated inner product, norm, and semi-norms in $H^s(D)$ are denoted by $(\cdot,\cdot)_{s,D}$, $\|\cdot\|_{s,D}$, and $|\cdot|_{s,D}$, respectively. When $s = 0$, $H^0(D)$ coincides with the space of square integrable functions $L^2(D)$. In this case, the subscript s is suppressed from the notation of norm, semi-norm, and inner products. Furthermore, the subscript D is also suppressed when $D = \Omega$.

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of polygons in two dimension or polyhedra in three dimension satisfying a set of conditions specified in [16] and additional conditions specified in Lemma 3.1. Denote by $\mathcal{E}_h$ the set of all edges/faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ be the set of all interior edges/faces. For simplicity, we will use term edge for edge/face without confusion.

For simplicity, we adopt the following notations,

$$\begin{aligned}
(v, w)_{\mathcal{T}_h} &= \sum_{T\in\mathcal{T}_h} (v, w)_T = \sum_{T\in\mathcal{T}_h} \int_T vw d\mathbf{x}, \\
\langle v, w\rangle_{\partial\mathcal{T}_h} &= \sum_{T\in\mathcal{T}_h} \langle v, w\rangle_{\partial T} = \sum_{T\in\mathcal{T}_h} \int_{\partial T} vw ds.
\end{aligned}$$

Let $P_k(K)$ consist all the polynomials degree less or equal to k defined on K.

Algorithm 1. *A conforming DG finite element method for the problem (1)-(2) seeks $u_h \in V_h$ satisfying*

$$(\nabla_w u_h, \nabla_w v)_{\mathcal{T}_h} \quad = \quad (f,\ v) \quad \forall v \in V_h. \tag{6}$$

The weak gradient ∇_w in the equation (6) is defined as follows [17, 10, 15, 16]. For a given $T \in \mathcal{T}_h$ and a function $v \in V_h + H_0^1(\Omega)$, the weak gradient $\nabla_w v \in [P_j(T)]^d$ on T satisfies the following equation,

$$(\nabla_w v, \mathbf{q})_T = -(v, \nabla \cdot \mathbf{q})_T + \langle \{v\}, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T} \qquad \forall \mathbf{q} \in [P_j(T)]^d, \tag{7}$$

where j and $\{v\}$ will be defined later.

In the following, we will introduce two finite element formulations by choosing the vector spaces V_h and the definition of average $\{\cdot\}$ differently.

Let T_1 and T_2 be two polygons/polyhedrons sharing e if $e \in \mathcal{E}_h^0$. For $e \in \mathcal{E}_h$ and $v \in V_h + H_0^1(\Omega)$, the jump $[v]$ is defined as

$$[v] = v \quad \text{if } e \subset \partial\Omega, \quad [v] = v|_{T_1} - v|_{T_2} \ \text{ if } e \in \mathcal{E}_h^0. \tag{8}$$

The order of T_1 and T_2 is not essential.

Case 1. Strongly enforce boundary condition

In this case, V_h is defined for $k \geq 1$ as

$$V_h = \left\{ v \in L^2(\Omega) : \ v|_T \in P_k(T) \ \ T \in \mathcal{T}_h, \quad v|_{\partial\Omega} = 0 \right\}. \tag{9}$$

For $e \in \mathcal{E}_h$ and $v \in V_h + H_0^1(\Omega)$, the average $\{v\}$ is defined as

$$\{v\} = v \quad \text{if } e \subset \partial\Omega, \quad \{v\} = \frac{1}{2}(v|_{T_1} + v|_{T_2}) \ \text{ if } e \in \mathcal{E}_h^0. \tag{10}$$

Case 2. Weakly enforce boundary condition

Here, V_h is defined for $k \geq 1$ as

$$V_h = \left\{ v \in L^2(\Omega) : \ v|_T \in P_k(T), \ \ T \in \mathcal{T}_h \right\}. \tag{11}$$

For $e \in \mathcal{E}_h$ and $v \in V_h + H_0^1(\Omega)$, the average $\{v\}$ is defined as

$$\{v\} = 0 \quad \text{if } e \subset \partial\Omega, \quad \{v\} = \frac{1}{2}(v|_{T_1} + v|_{T_2}) \ \text{ if } e \in \mathcal{E}_h^0. \tag{12}$$

Remark 1. *For the finite element formulation (6) associated with Case 1, we assume that each element $T \in \mathcal{T}_h$ has no more than two edges on $\partial\Omega$ in 2D, or no more than 3 faces on $\partial\Omega$ in 3D. This requirement is only needed for error analysis. In practice, we cannot find any meshes consisting of elements sharing more than two edges in 2D and three faces in 3D with $\partial\Omega$ after any mesh refinement.*

Lemma 2.1. *Let $\phi \in H_0^1(\Omega)$, then on $T \in \mathcal{T}_h$*

$$\nabla_w \phi = \mathbb{Q}_h \nabla \phi. \tag{13}$$

Proof. Using (7) and integration by parts, we have that for any $\mathbf{q} \in [P_j(T)]^d$

$$\begin{aligned} (\nabla_w \phi, \mathbf{q})_T &= -(\phi, \nabla \cdot \mathbf{q})_T + \langle \{\phi\}, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T} \\ &= -(\phi, \nabla \cdot \mathbf{q})_T + \langle \phi, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T} \\ &= (\nabla \phi, \mathbf{q})_T = (\mathbb{Q}_h \nabla \phi, \mathbf{q})_T, \end{aligned}$$

which implies the desired identity (13). $\square$

3. Well Posedness

We start this section by introducing a semi-norms $|||v|||$ and a norm $\|v\|_{1,h}$ for any $v \in V_h + H_0^1(\Omega)$ as follows:

$$|||v|||^2 = \sum_{T\in\mathcal{T}_h} (\nabla_w v, \nabla_w v)_T, \tag{14}$$

$$\|v\|_{1,h}^2 = \sum_{T\in\mathcal{T}_h} \|\nabla v\|_T^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[v]\|_e^2. \tag{15}$$

For any function $\varphi \in H^1(T)$, the following trace inequality holds true (see [16] for details):

$$\|\varphi\|_e^2 \le C\left(h_T^{-1}\|\varphi\|_T^2 + h_T\|\nabla\varphi\|_T^2\right). \tag{16}$$

Lemma 3.1. *Let T be a convex $(n+1)$-polygon/polyhedron of size h_T with edges/faces e, e_1, ..., and e_n, satisfying minor angle and length conditions to be specified in the proof below. For a given polynomial $q_0 \in P_k(e)$, we define a polynomial $q \in P_{k+n}(T)$ by*

$$q = \lambda_1 \cdots \lambda_n q_1, \quad \textit{where } q_1 \in P_k(T) \textit{ satisfying} \tag{17}$$

$$\langle q - q_0, p\rangle_e = 0 \quad \forall p \in P_k(e), \tag{18}$$

$$(q, p)_T = 0 \quad \forall p \in P_{k-1}(T), \tag{19}$$

where $\lambda_i \in P_1(T)$ vanishes on e_i and assumes value 1 at the barycenter of e. Then it holds that

$$\|q\|_T \le C h_T^{1/2}\|q_0\|_e, \tag{20}$$

where the nonzero constant is defined in (26) *below, independent of T and q_0.*

Proof. First the linear system (18)–(19) of equation is square, of size $\dim P_k$. To show its existence and uniqueness of solution, we need only to show the uniqueness. Let $q_0 = 0$ and $p = q_1$ in (18). It follows that $q_1 \equiv 0$ on e and $q_1 = \lambda_0 q_2$ for some $q_2 \in P_{k-1}(T)$ because the weight is positive in the weighted $L^2(e)$ inner product. Here $\lambda_0 \in P_1(T)$, $\lambda_0|_e = 0$, and $\max_T \lambda_0 = 1$. Next letting $p = q_2$ in (19), due to a positive weight $\prod_{i=0}^n \lambda_i$ on T^0, we have $q_2 = 0$.

If e_i is a neighboring edge/face of e, then

$$\lambda_i|_e = \frac{2}{h_e}x$$

where h_e is the doubled distance from the barycenter of e to e_i along/on e and x is the distance from a point on e to e_i along (2D) or on (3D) e. For simplicity, we assume this h_e is also the size of e (it is indeed in 2D). To avoid too many constants, we assume $h_e \ge h_T/4$. Then

$$\max_T \lambda_i = \frac{h_{\perp e_i}(T)}{(h_e/2)\sin\alpha_i} \le \frac{h_T}{(h_e/2)\sin\alpha_i} \le \frac{8}{\sin\alpha_i} \le \frac{8}{\sin\alpha_0}, \tag{21}$$

where $\pi - \alpha_i$ (for some $\alpha_i \ge \alpha_0 > 0$ and $\alpha_i \le \pi - \alpha_0$) is the angle between e and e_i, $h_{\perp e_i}(T)$ is the maximal distance of points on T to e_i in the direction orthogonal to e_i. Let $e_1, \ldots, e_m$ are all the neighboring edges/faces of e, $m = 2$ in 2D, and $m \le n$. For a lower bound, we have

$$\lambda_i|_{T_0} \ge \begin{cases} \frac{15}{16} & \text{if } \alpha_i \le \pi/2, \\ 1 - \frac{\sqrt{d}}{16\sin\alpha_i} \ge \frac{1}{2} & \text{if } \alpha_i > \pi/2, \end{cases} \tag{22}$$

where T_0 is a square/cube at middle of e with size $h_e/16$, cf. Figure 1. We note that other than triangles, $\alpha_i \le \pi/2$ for most other polygons. Here in (22), we assumed $\sin\alpha_0 \ge \sqrt{d}/8$, where d is the space dimension, 2 or 3.

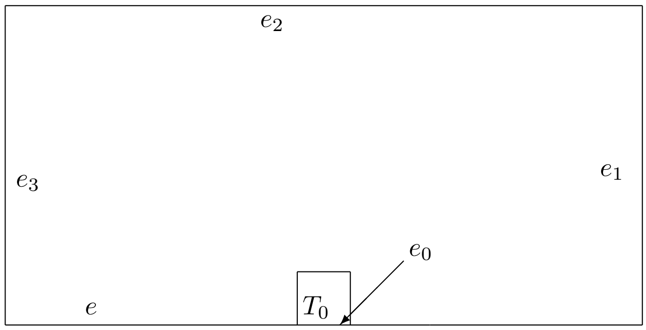

FIGURE 1. Size $|e_0| = |e|/16 = e_h/8$, and T_0 is square of size $|e_0|$.

For non-neighboring edges e_j, we have

$$\lambda_j|_{e_1} = \begin{cases} 1 & \text{if } e_j \parallel e_1, \\ \frac{2(x+x_j)}{h_{e_1}+x_j} & \text{otherwise,} \end{cases}$$

where x is the arc-length parametrization on e toward the extended intersection of e and e_i, x_j is the distance on e from the an boundary point of e to the intersection. Supposing e_i is the only edge/polygonal between e and e_j, $x_j = h_{e_i}(\cos\alpha_i - \cos(\alpha_i + \alpha_j))$. Because $x_j \ge 0$, it follows that

$$\max_T \lambda_j = \frac{h_{\perp e_j}(T)}{(h_e/2)\sin\alpha_i} \le \frac{2h_T}{(h_e + x_j)\sin(\alpha_i+\alpha_j)} \le \frac{8}{\sin\alpha_0}. \tag{23}$$

For a lower bound, because $x_j > 0$ and e_i is an edge/polygon in between, we have

$$\lambda_j|_{T_0} \ge \lambda_i|_{T_0} \ge \frac{1}{2}. \tag{24}$$

Together, we have, noting $\lambda_0|_T \le 1$,

$$\lambda_1 \cdots \lambda_n|_{T_0} \ge \frac{1}{2^n}, \quad \text{and } \lambda_0\lambda_1\cdots\lambda_n|_T \le \frac{8^n}{\sin^n\alpha_0}. \tag{25}$$

Let $\tilde q_1 \in P_k(e)$ be the solution in (18). Letting $\tilde p = q_1$ in (18), by (25), we get

$$\begin{aligned} \frac{1}{16^{2k}}\frac{1}{2^n}\|\tilde q_1\|_e^2 \le \frac{1}{2^n}\|\tilde q_1\|_{e_0}^2 &\le \langle \lambda_1\cdots\lambda_n \tilde q_1, \tilde q_1\rangle_e \\ &= \langle q_0, \tilde q_1\rangle_e \le \|q_0\|_0\|\tilde q_1\|_0, \end{aligned}$$

where in the first step we use the fact q_1 is a degree k polynomial. We view $\tilde q_1 \in P_k(e)$ as defined on the whole line/plane passing through e. We extend this polynomial to a polynomial $\tilde q_1$ in $P_k(\mathbb{R}^d)$, by letting it be constant in the direction orthogonal to e. In particular, we have, as $T \subset S_T$ and $e \subset S_e$,

$$\begin{aligned} \|\tilde q_1\|_T^2 \le \|\tilde q_1\|_{S_T}^2 = h_T\|\tilde q_1\|_{S_e}^2 &\le \left(\frac{h_T}{h_e}\right)^{2k} h_T\|\tilde q_1\|_e^2 \\ &\le 4^{2k}h_T\|\tilde q_1\|_e^2 \le 2^{4k}h_T(2^{8k+n}\|q_0\|_e)^2, \end{aligned}$$

where S_T is a square/cube of size h_T containing T, with one side S_e which contains e.

Rewriting (17) in terms of this extended $\tilde{q}_1$, we have

$$q = \lambda_1 \cdots \lambda_n(\lambda_0 q_2 + \tilde{q}_1)$$

for some $q_2 \in P_{k-1}(T)$. Letting $p = q_2$ in (19), by (25), we have

$$\begin{aligned}
\|q_2\|_T^2 \le (h_T/h_{e_0})^{2k-2}\,\|q_2\|_{T_0}^2 &\le 64^{2k-2}\frac{8^n}{\sin^n\alpha_0}(\lambda_1\cdots\lambda_n q_2, q_2)_{T_0}\\
&\le \frac{2^{3n+12k-12}}{\sin^n\alpha_0}\frac{2h_T}{h_{e_0}}(\lambda_1\cdots\lambda_n\lambda_0 q_2, q_2)_{T_{0,0}}\\
&\le \frac{2^{3n+12k-5}}{\sin^n\alpha_0}(\lambda_1\cdots\lambda_n\lambda_0 q_2, q_2)_T\\
&= \frac{2^{3n+12k-5}}{\sin^n\alpha_0}(\lambda_1\cdots\lambda_n\tilde{q}_1, -q_2)_T\\
&\le \frac{2^{3n+12k-5}}{\sin^n\alpha_0}2^n\|\tilde{q}_1\|_T\|q_2\|_T,
\end{aligned}$$

where $T_{0,0}$ is the top half of T_0, cf. Figure 1. Then,

$$\begin{aligned}
\|q\|_T^2 &= (\lambda_1^2\cdots\lambda_n^2(\lambda_0 q_2 - \tilde{q}_1), (\lambda_0 q_2 - \tilde{q}_1))_T\\
&\le \frac{8^{2n}}{\sin^{2n}\alpha_0}((\lambda_0 q_2 - \tilde{q}_1), (\lambda_0 q_2 - \tilde{q}_1))_T\\
&\le \frac{8^{2n}}{\sin^{2n}\alpha_0}2(\|\lambda_0 q_2\|_T^2 + \|\tilde{q}_1\|_T^2)\\
&\le \frac{2^{6n+1}}{\sin^{2n}\alpha_0}(\|q_2\|_T^2 + \|\tilde{q}_1\|_T^2),
\end{aligned}$$

where $\lambda_0 \le 1$ on T. Finally, combining above three bounds, we get

$$\begin{aligned}
\|q\|_T &\le \frac{2^{3n+1/2}}{\sin^n\alpha_0}\left((\frac{2^{4n+12k-5}}{\sin^n\alpha_0})^2+1\right)^{\frac12}\|\tilde{q}_1\|_T\\
&\le \frac{2^{10k+4n+1/2}}{\sin^n\alpha_0}\left((\frac{2^{4n+12k-5}}{\sin^n\alpha_0})^2+1\right)^{\frac12} h_T^{1/2}\|q_0\|_e \qquad (26)\\
&=: C h_T^{1/2}\|q_0\|_e.
\end{aligned}$$

The proof is completed. □

Lemma 3.2. *There exist two positive constants C_1 and C_2 independent of mesh size h such that for any $v \in V_h$, we have*

$$C_1\|v\|_{1,h} \le |||v||| \le C_2\|v\|_{1,h}. \qquad (27)$$

Proof. For any $v \in V_h$, it follows from the definition of weak gradient (7) and integration by parts that for all $\mathbf{q} \in [P_j(T)]^d$

$$\begin{aligned}
(\nabla_w v, \mathbf{q})_T &= -(v, \nabla\cdot\mathbf{q})_T + \langle\{v\}, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T}\\
&= (\nabla v, \mathbf{q})_T - \langle v - \{v\}, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T}. \qquad (28)
\end{aligned}$$

By letting $\mathbf{q} = \nabla_w v$ in (28) we arrive at

$$(\nabla_w v, \nabla_w v)_T = (\nabla v, \nabla_w v)_T - \langle v - \{v\}, \nabla_w v\cdot\mathbf{n}\rangle_{\partial T}.$$

It is easy to see that the following equations hold true for $\{v\}$ defined in both (10) and (12),

$$\|v-\{v\}\|_e = \|[v]\|_e \quad \text{if } e\subset\partial\Omega, \quad \|v-\{v\}\|_e = \frac{1}{2}\|[v]\|_e \ \text{ if } e\in\mathcal{E}_h^0. \tag{29}$$

From (29), (16) and the inverse inequality we have

$$\begin{aligned}
\|\nabla_w v\|_T^2 &\le \|\nabla v\|_T\|\nabla_w v\|_T + \|v-\{v\}\|_{\partial T}\|\nabla_w v\|_{\partial T}\\
&\le \|\nabla v\|_T\|\nabla_w v\|_T + Ch_T^{-1/2}\|v-\{v\}\|_{\partial T}\|\nabla_w v\|_T\\
&\le \|\nabla v\|_T\|\nabla_w v\|_T + Ch_T^{-1/2}\|[v]\|_{\partial T}\|\nabla_w v\|_T
\end{aligned}$$

which implies

$$\|\nabla_w v\|_T \le C\left(\|\nabla v\|_T + Ch_T^{-1/2}\|[v]\|_{\partial T}\right),$$

and consequently

$$|\!|\!|v|\!|\!| \le C_2\|v\|_{1,h}.$$

Next we will prove $C_1\|v\|_{1,h}\le |\!|\!|v|\!|\!|$. For $v\in V_h$ and $\mathbf{q}\in[P_j(T)]^d$, by (7) and integration by parts, we have

$$(\nabla_w v,\mathbf{q})_T = (\nabla v,\mathbf{q})_T + \langle\{v\}-v,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T}. \tag{30}$$

We like to find $\mathbf{q}_0\in[P_j(T)]^d$ such that,

$$(\nabla v,\mathbf{q}_0)_T=0,\quad \langle\{v\}-v,\mathbf{q}_0\cdot\mathbf{n}\rangle_{\partial T\setminus e}=0,\ \text{ and }\ \langle\{v\}-v,\mathbf{q}_0\cdot\mathbf{n}\rangle_e=\|\{v\}-v\|_e^2, \tag{31}$$

and

$$\|\mathbf{q}_0\|_T\le Ch_T^{1/2}\|\{v\}-v\|_e. \tag{32}$$

Letting $q_0=\{v\}-v$ in (18), there exists a $q\in P_{n+k-1}(T)$ (i.e. $j=n+k-1$) such that (18)–(20) hold, where n is the number of the edges/faces on a polygon/polyhadron. Without loss of generality, let $\mathbf{n}=\langle n_1,\cdots,n_d\rangle$ for some $n_1\neq 0$. We then let $\mathbf{q}_0=\langle q/n_1,0,\cdots,0\rangle$, which satisfies (31) and (32) by Lemma 3.1. Substituting $\mathbf{q}_0$ into (30), we get

$$(\nabla_w v,\mathbf{q}_0)_T=\|\{v\}-v\|_e^2. \tag{33}$$

It follows from Cauchy-Schwarz inequality that

$$\|\{v\}-v\|_e^2\le C\|\nabla_w v\|_T\|\mathbf{q}_0\|_T\le Ch_T^{1/2}\|\nabla_w v\|_T\|\{v\}-v\|_e,$$

which gives

$$h_T^{-1/2}\|\{v\}-v\|_{\partial T}\le C\|\nabla_w v\|_T. \tag{34}$$

Using (29) and summing the both sides of (34) over T, we obtain

$$\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[v]\|_e^2\le C|\!|\!|v|\!|\!|^2. \tag{35}$$

It follows from the trace inequality, the inverse inequality and (34),

$$\|\nabla v\|_T^2\le\|\nabla_w v\|_T\|\nabla v\|_T + Ch_T^{-1/2}\|\{v\}-v\|_{\partial T}\|\nabla v\|_T\le C\|\nabla_w v\|_T\|\nabla v\|_T,$$

which implies

$$\sum_{T\in\mathcal{T}_h}\|\nabla v\|_T^2\le C|\!|\!|v|\!|\!|^2. \tag{36}$$

Combining (35) and (36), we prove the lower bound of (27) and complete the proof of the lemma. $\square$

4. Error Estimates in Energy Norm

We start this section by defining some approximation operators. Let $\mathbb{Q}_h$ be the element-wise defined L^2 projection onto $[P_j(T)]^d$ on each element T. We will call any element $T \in \mathcal{T}_h$, that has one or two edges on $\partial\Omega$, boundary element in 2D. Then we will define $I_h u$, an interpolation of u, on boundary elements. $I_h u$ for 3D can be constructed in a similar fashion. For a boundary element T, let $T_0 \subset T$ be a triangle such that $\partial T \cap \partial\Omega = \partial T_0 \cap \partial\Omega$. Let $I_h u$ be kth order interpolation of u on T_0.

Lemma 4.1. *For any boundary element $T \in \mathcal{T}_h$, one has*

$$\|u - I_h u\|_T + h_T\|\nabla(u - I_h u)\|_T \le Ch^{k+1}|u|_{k+1,T}. \tag{37}$$

Proof. For any boundary element $T \in \mathcal{T}_h$, by the construction of $I_h u$, one has

$$\|u - I_h u\|_{T_0} + h_T\|\nabla(u - I_h u)\|_{T_0} \le Ch^{k+1}|u|_{k+1,T_0}. \tag{38}$$

Let Q_0 be the L^2 projection onto $P_k(T)$. The following estimate holds [9]

$$\|u - Q_0 u\|_T + h_T\|\nabla(u - Q_0 u)\|_T \le Ch^{k+1}|u|_{k+1,T}. \tag{39}$$

By the triangle inequality, then

$$\|u - I_h u\|_T \le \|u - Q_0 u\|_T + \|Q_0 u - I_h u\|_T. \tag{40}$$

By the domain inverse inequality [11, 12] and under necessary regularity assumption of the mesh $\mathcal{T}_h$, we have

$$\|Q_0 u - I_h u\|_T \le C\|Q_0 u - I_h u\|_{T_0} \le C(\|Q_0 u - u\|_{T_0} + \|u - I_h u\|_{T_0}). \tag{41}$$

Combining (38)-(41) yields

$$\|u - I_h u\|_T \le Ch^{k+1}|u|_{k+1,T}.$$

Similarly, we can prove the second part of the estimate in (37) and finish the proof of the lemma. □

Now we define $Q_h u \in V_h$, an approximation of u for the two finite element methods associated with Case 1 and Case 2. For the method associated with Case 1, let $Q_h u = Q_0 u$ for any T which is not boundary element and $Q_h u = I_h u$ for the boundary element T. For the case 2, define $Q_h u = Q_0 u$ for all $T \in \mathcal{T}_h$.

Let $e_h = u - u_h$ and $\epsilon_h = Q_h u - u_h \in V_h$. Next we derive an error equation that e_h satisfies.

Lemma 4.2. *For any $v \in V_h$, one has,*

$$(\nabla_w e_h, \nabla_w v)_{\mathcal{T}_h} = \ell(u, v), \tag{42}$$

where

$$\ell(u,v) \quad = \quad \langle(\nabla u - \mathbb{Q}_h\nabla u)\cdot\mathbf{n}, v - \{v\}\rangle_{\partial T_h}.$$

Proof. Testing (1) by any $v \in V_h$ and using integration by parts and the fact that $\sum_{T\in\mathcal{T}_h}\langle\nabla u\cdot\mathbf{n}, \{v\}\rangle_{\partial T} = 0$ for $\{v\}$ defined in both (10) and (12), we arrive at

$$(\nabla u, \nabla v)_{\mathcal{T}_h} - \langle\nabla u\cdot\mathbf{n}, v - \{v\}\rangle_{\partial T_h} = (f, v). \tag{43}$$

It follows from integration by parts, (7) and (13) that

$$\begin{aligned}
(\nabla u, \nabla v)_{\mathcal{T}_h} &= (\mathbb{Q}_h\nabla u, \nabla v)_{\mathcal{T}_h} \\
&= -(v, \nabla\cdot(\mathbb{Q}_h\nabla u))_{\mathcal{T}_h} + \langle v, \mathbb{Q}_h\nabla u\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} \\
&= (\mathbb{Q}_h\nabla u, \nabla_w v)_{\mathcal{T}_h} + \langle v - \{v\}, \mathbb{Q}_h\nabla u\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} \\
&= (\nabla_w u, \nabla_w v)_{\mathcal{T}_h} + \langle v - \{v\}, \mathbb{Q}_h\nabla u\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}.
\end{aligned} \tag{44}$$

Combining (43) and (44) gives

$$(\nabla_w u, \nabla_w v)_{\mathcal{T}_h} \quad = \quad (f, v) + \ell(u, v). \tag{45}$$

The error equation follows from subtracting (6) from (45),

$$(\nabla_w e_h, \nabla_w v)_{\mathcal{T}_h} = \ell(u, v) \quad \forall v \in V_h.$$

This completes the proof of the lemma. □

Lemma 4.3. *For any $w \in H^{k+1}(\Omega)$ and $v \in V_h$, we have*

$$|\ell(w, v)| \quad \leq \quad Ch^k |w|_{k+1} |||v|||. \tag{46}$$

Proof. Using the Cauchy-Schwarz inequality, the trace inequality (16), (29) and (27), we have

$$\begin{aligned}
|\ell(w, v)| &= \left| \sum_{T \in \mathcal{T}_h} \langle (\nabla w - \mathbb{Q}_h \nabla w) \cdot \mathbf{n}, v - \{v\} \rangle_{\partial T} \right| \\
&\leq C \sum_{T \in \mathcal{T}_h} \|\nabla w - \mathbb{Q}_h \nabla w\|_{\partial T} \|v - \{v\}\|_{\partial T} \\
&\leq C \left(\sum_{T \in \mathcal{T}_h} h_T \|(\nabla w - \mathbb{Q}_h \nabla w)\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{e \in \mathcal{E}_h} h_e^{-1} \|[v]\|_e^2 \right)^{\frac{1}{2}} \\
&\leq Ch^k |w|_{k+1} |||v|||,
\end{aligned}$$

which proves the lemma. □

Lemma 4.4. *Let $u \in H^{k+1}(\Omega)$, then*

$$|||u - Q_h u||| \leq Ch^k |u|_{k+1}. \tag{47}$$

Proof. It follows from (7), integration by parts, (16) and (29),

$$\begin{aligned}
|(\nabla_w (u - Q_h u), \mathbf{q})_T| &= | - (u - Q_h u, \nabla \cdot \mathbf{q})_T + \langle u - \{Q_h u\}, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T}| \\
&= |(\nabla (u - Q_h u), \mathbf{q})_T + \langle Q_h u - \{Q_h u\}, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T}| \\
&\leq \|\nabla (u - Q_h u)\|_T \|\mathbf{q}\|_T + Ch^{-1/2} \|[Q_h u]\|_{\partial T} \|\mathbf{q}\|_T \\
&\leq \|\nabla (u - Q_h u)\|_T \|\mathbf{q}\|_T + Ch^{-1/2} \|[u - Q_h u]\|_{\partial T} \|\mathbf{q}\|_T \\
&\leq Ch^k |u|_{k+1,T} \|\mathbf{q}\|_T.
\end{aligned}$$

Letting $\mathbf{q} = \nabla_w (u - Q_h u)$ in the above equation and taking summation over T, we have

$$|||u - Q_h u||| \leq Ch^k |u|_{k+1}.$$

We have proved the lemma. □

Theorem 4.1. *Let $u_h \in V_h$ be the finite element solution of (6). Assume the exact solution $u \in H^{k+1}(\Omega)$. Then, there exists a constant C such that*

$$|||u - u_h||| \leq Ch^k |u|_{k+1}. \tag{48}$$

Proof. It is straightforward to obtain

$$\begin{aligned}
|||e_h|||^2 &= (\nabla_w e_h, \nabla_w e_h)_{\mathcal{T}_h} \\
&= (\nabla_w u - \nabla_w u_h, \nabla_w e_h)_{\mathcal{T}_h} \\
&= (\nabla_w Q_h u - \nabla_w u_h, \nabla_w e_h)_{\mathcal{T}_h} + (\nabla_w u - \nabla_w Q_h u, \nabla_w e_h)_{\mathcal{T}_h} \\
&= (\nabla_w e_h, \nabla_w \epsilon_h)_{\mathcal{T}_h} + (\nabla_w (u - Q_h u), \nabla_w e_h)_{\mathcal{T}_h}.
\end{aligned} \tag{49}$$

We will bound each terms in (49). Letting $v = \epsilon_h \in V_h$ in (42) and using (46) and (47), we have

$$\begin{aligned} |(\nabla_w e_h, \nabla_w \epsilon_h)_{\mathcal{T}_h}| &= |\ell(u, \epsilon_h)| \\ &\le Ch^k |u|_{k+1} |\!|\!| \epsilon_h |\!|\!| \\ &\le Ch^k |u|_{k+1} |\!|\!| Q_h u - u_h |\!|\!| \\ &\le Ch^k |u|_{k+1} (|\!|\!| Q_h u - u |\!|\!| + |\!|\!| u - u_h |\!|\!|) \\ &\le Ch^{2k} |u|_{k+1}^2 + \frac{1}{4} |\!|\!| e_h |\!|\!|^2. \end{aligned} \tag{50}$$

The estimate (47) implies

$$\begin{aligned} |(\nabla_w (u - Q_h u), \nabla_w e_h)_{\mathcal{T}_h}| &\le C |\!|\!| u - Q_h u |\!|\!| |\!|\!| e_h |\!|\!| \\ &\le Ch^{2k} |u|_{k+1}^2 + \frac{1}{4} |\!|\!| e_h |\!|\!|^2. \end{aligned} \tag{51}$$

Combining the estimates (50) and (51) with (49), we arrive

$$|\!|\!| e_h |\!|\!| \le Ch^k |u|_{k+1},$$

which completes the proof. □

5. Error Estimates in L^2 Norm

The standard duality argument is used to obtain L^2 error estimate. Recall $e_h = u - u_h$ and $\epsilon_h = Q_h u - u_h$. The considered dual problem seeks $\Phi \in H_0^1(\Omega)$ satisfying

$$-\Delta \Phi = e_h \quad \text{in } \Omega. \tag{52}$$

Assume that the following H^2-regularity holds

$$\|\Phi\|_2 \le C \|e_h\|. \tag{53}$$

Theorem 5.1. *Let $u_h \in V_h$ be the finite element solution of (6). Assume that the exact solution $u \in H^{k+1}(\Omega)$ and (53) holds true. Then, there exists a constant C such that*

$$\|u - u_h\| \le Ch^{k+1} |u|_{k+1}. \tag{54}$$

Proof. Testing (52) by e_h and using the fact that $\sum_{T \in \mathcal{T}_h} \langle \nabla \Phi \cdot \mathbf{n}, \{e_h\} \rangle_{\partial T} = 0$ and (7) give

$$\begin{aligned} \|e_h\|^2 &= -(\Delta \Phi, e_h) \\ &= (\nabla \Phi,\ \nabla e_h)_{\mathcal{T}_h} - \langle \nabla \Phi \cdot \mathbf{n},\ e_h - \{e_h\} \rangle_{\partial T_h} \\ &= (\mathbb{Q}_h \nabla \Phi,\ \nabla e_h)_{\mathcal{T}_h} + (\nabla \Phi - \mathbb{Q}_h \nabla \Phi,\ \nabla e_h)_{\mathcal{T}_h} - \langle \nabla \Phi \cdot \mathbf{n},\ e_h - \{e_h\} \rangle_{\partial T_h} \\ &= -(\nabla \cdot \mathbb{Q}_h \nabla \Phi,\ e_h)_{\mathcal{T}_h} + \langle \mathbb{Q}_h \nabla \Phi \cdot \mathbf{n},\ e_h \rangle_{\partial T_h} \\ &+ (\nabla \Phi - \mathbb{Q}_h \nabla \Phi,\ \nabla e_h)_{\mathcal{T}_h} - \langle \nabla \Phi \cdot \mathbf{n},\ e_h - \{e_h\} \rangle_{\partial T_h} \\ &= (\mathbb{Q}_h \nabla \Phi,\ \nabla_w e_h)_{\mathcal{T}_h} + \langle \mathbb{Q}_h \nabla \Phi \cdot \mathbf{n},\ e_h - \{e_h\} \rangle_{\partial T_h} \\ &+ (\nabla \Phi - \mathbb{Q}_h \nabla \Phi,\ \nabla e_h)_{\mathcal{T}_h} - \langle \nabla \Phi \cdot \mathbf{n},\ e_h - \{e_h\} \rangle_{\partial T_h} \\ &= (\mathbb{Q}_h \nabla \Phi,\ \nabla_w e_h)_{\mathcal{T}_h} + (\nabla \Phi - \mathbb{Q}_h \nabla \Phi,\ \nabla e_h)_{\mathcal{T}_h} - \ell(\Phi, e_h). \end{aligned}$$

It follows from (13) and (42)

$$\begin{aligned} (\mathbb{Q}_h \nabla \Phi,\ \nabla_w e_h)_{\mathcal{T}_h} &= (\nabla_w \Phi,\ \nabla_w e_h)_{\mathcal{T}_h} \\ &= (\nabla_w Q_h \Phi,\ \nabla_w e_h)_{\mathcal{T}_h} + (\nabla_w (\Phi - Q_h \Phi),\ \nabla_w e_h)_{\mathcal{T}_h} \\ &= \ell(u, Q_h \Phi) + (\nabla_w (\Phi - Q_h \Phi),\ \nabla_w e_h)_{\mathcal{T}_h}. \end{aligned}$$

Combining the two equations above gives

$$\begin{aligned}\|e_h\|^2 &= \ell(u, Q_h\Phi) + (\nabla_w(\Phi - Q_h\Phi),\ \nabla_w e_h)_{\mathcal{T}_h} \\ &+ (\nabla\Phi - \mathbb{Q}_h\nabla\Phi,\ \nabla e_h)_{\mathcal{T}_h} + \ell(\Phi, e_h).\end{aligned} \tag{55}$$

Next we will estimate all the terms on the right hand side of (55). Using the Cauchy-Schwarz inequality, the trace inequality (16) and the definitions of Q_h and $\mathbb{Q}_h$ we obtain

$$\begin{aligned}|\ell(u, Q_h\Phi)| &\le |\langle(\nabla u - \mathbb{Q}_h\nabla u)\cdot\mathbf{n},\ Q_h\Phi - \{Q_h\Phi\}\rangle_{\partial T_h}| \\ &\le \left(\sum_{T\in\mathcal{T}_h}\|(\nabla u - \mathbb{Q}_h\nabla u)\|_{\partial T}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}\|Q_h\Phi - \{Q_h\Phi\}\|_{\partial T}^2\right)^{1/2} \\ &\le C\left(\sum_{T\in\mathcal{T}_h} h\|(\nabla u - \mathbb{Q}_h\nabla u)\|_{\partial T}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h} h^{-1}\|[Q_h\Phi - \Phi]\|_{\partial T}^2\right)^{1/2} \\ &\le Ch^{k+1}|u|_{k+1}|\Phi|_2.\end{aligned}$$

It follows from (48) and (47) that

$$\begin{aligned}|(\nabla_w e_h,\ \nabla_w(\Phi - Q_h\Phi))_{\mathcal{T}_h}| &\le C|\!|\!|e_h|\!|\!||\!|\!|\Phi - Q_h\Phi|\!|\!| \\ &\le Ch^{k+1}|u|_{k+1}|\Phi|_2.\end{aligned}$$

The norm equivalence (27) implies

$$\begin{aligned}|(\nabla\Phi - \mathbb{Q}_h\nabla\Phi,\ \nabla e_h)_{\mathcal{T}_h}| &\le C(\sum_{T\in\mathcal{T}_h}\|\nabla e_h\|_T^2)^{1/2}(\sum_{T\in\mathcal{T}_h}\|\nabla\Phi - \mathbb{Q}_h\nabla\Phi\|_T^2)^{1/2} \\ &\le C(\sum_{T\in\mathcal{T}_h}(\|\nabla(u - Q_h u)\|_T^2 + \|\nabla(Q_h u - u_h)\|_T^2))^{1/2} \\ &\times (\sum_{T\in\mathcal{T}_h}\|\nabla\Phi - \mathbb{Q}_h\nabla\Phi\|_T^2)^{1/2} \\ &\le Ch|\Phi|_2(h^k|u|_{k+1} + |\!|\!|Q_h u - u_h|\!|\!|) \\ &\le Ch|\Phi|_2(h^k|u|_{k+1} + |\!|\!|u - u_h|\!|\!| + |\!|\!|Q_h u - u|\!|\!|) \\ &\le Ch^{k+1}|u|_{k+1}|\Phi|_2.\end{aligned}$$

Using (27), (29), (48), and (47), we obtain

$$\begin{aligned}|\ell(\Phi, e_h)| &= \left|\sum_{T\in\mathcal{T}_h}\langle(\mathbb{Q}_h\nabla\Phi - \nabla\Phi)\cdot\mathbf{n},\ e_h - \{e_h\}\rangle_{\partial T}\right| \\ &\le \sum_{T\in\mathcal{T}_h} h_T^{1/2}\|\mathbb{Q}_h\nabla\Phi - \nabla\Phi\|_{\partial T} h_T^{-1/2}\|[e_h]\|_{\partial T} \\ &\le Ch\|\Phi\|_2(\sum_{T\in\mathcal{T}_h} h_T^{-1}(\|[\varepsilon_h]\|_{\partial T}^2 + \|[u - Q_h u]\|_{\partial T}^2)^{1/2} \\ &\le Ch\|\Phi\|_2(|\!|\!|\varepsilon_h|\!|\!| + (\sum_{T\in\mathcal{T}_h} h_T^{-1}\|[u - Q_h u]\|_{\partial T}^2)^{1/2} \\ &\le Ch\|\Phi\|_2(|\!|\!|e_h|\!|\!| + |\!|\!|u - Q_h u|\!|\!| + Ch^k|u|_{k+1}) \\ &\le Ch^{k+1}|u|_{k+1}\|\Phi\|_2.\end{aligned}$$

Combining all the estimates above with (55) yields

$$\|e_h\|^2 \le Ch^{k+1}|u|_{k+1}\|\Phi\|_2.$$

The estimate (54) follows from the above inequality and the regularity assumption (53). We have completed the proof. □

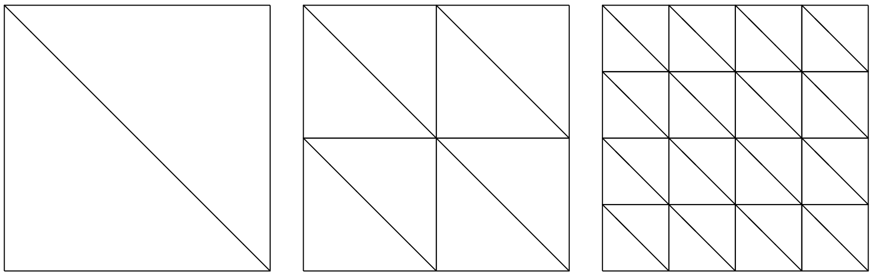

FIGURE 2. The first three levels of grids used in the computation of Table 1.

TABLE 1. Error profiles and convergence rates for (56) on triangular grids (Figure 2).

level	$\|u_h - Q_0 u\|_0$	rate	$\|\|\|u_h - u\|\|\|$	rate	dim
	by P_1 elements with strongly enforced boundary condition				
6	0.5655E-03	2.00	0.8945E-01	1.00	5890
7	0.1412E-03	2.00	0.4463E-01	1.00	24066
8	0.3526E-04	2.00	0.2229E-01	1.00	97282
	by P_1 elements with weakly enforced boundary condition				
6	0.5970E-03	2.09	0.8575E-01	0.94	6144
7	0.1449E-03	2.04	0.4371E-01	0.97	24576
8	0.3570E-04	2.02	0.2206E-01	0.99	98304
	by P_2 elements with strongly enforced boundary condition				
6	0.6635E-05	2.99	0.1797E-02	2.00	11906
7	0.8314E-06	3.00	0.4489E-03	2.00	48386
8	0.1040E-06	3.00	0.1122E-03	2.00	195074
	by P_2 elements with weakly enforced boundary condition				
6	0.6446E-05	2.94	0.1744E-02	1.95	12288
7	0.8197E-06	2.98	0.4424E-03	1.98	49152
8	0.1033E-06	2.99	0.1113E-03	1.99	196608
	by P_3 elements with strongly enforced boundary condition				
6	0.4263E-07	4.00	0.2253E-04	3.01	19970
7	0.2664E-08	4.00	0.2810E-05	3.00	80898
8	0.1666E-09	4.00	0.3509E-06	3.00	325634
	by P_3 elements with weakly enforced boundary condition				
6	0.4311E-07	4.02	0.2193E-04	2.97	20480
7	0.2679E-08	4.01	0.2772E-05	2.98	81920
8	0.1670E-09	4.00	0.3485E-06	2.99	327680

TABLE 2. Error profiles and convergence rates for (56) on triangular grids (Figure 2).

level	$\|u_h - Q_0 u\|_0$	rate	$\|\|\|u_h - u\|\|\|$	rate	dim
	by P_4 elements with strongly enforced boundary condition				
4	0.6433E-06	4.96	0.7511E-04	3.98	1762
5	0.2021E-07	4.99	0.4699E-05	4.00	7362
6	0.6320E-09	5.00	0.2934E-06	4.00	30082
	by P_4 elements with weakly enforced boundary condition				
4	0.6781E-06	5.03	0.7116E-04	3.90	1920
5	0.2076E-07	5.03	0.4577E-05	3.96	7680
6	0.6407E-09	5.02	0.2896E-06	3.98	30720
	by P_5 elements with strongly enforced boundary condition				
4	0.2306E-07	5.94	0.3385E-05	5.01	2498
5	0.3668E-09	5.97	0.1050E-06	5.01	10370
6	0.5825E-11	5.98	0.3266E-08	5.01	42242
	by P_5 elements with weakly enforced boundary condition				
4	0.2481E-07	6.04	0.3223E-05	4.94	2688
5	0.3811E-09	6.02	0.1024E-06	4.98	10752
6	0.5938E-11	6.00	0.3225E-08	4.99	43008

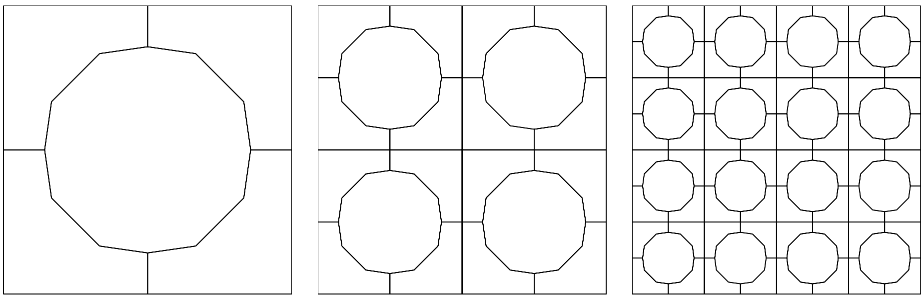

FIGURE 3. The first three polygonal grids for the computation of Table 3.

6. Numerical Example

We solve the following Poisson equation on the unit square:

$$-\Delta u = 2\pi^2 \sin \pi x \sin \pi y, \quad (x, y) \in \Omega = (0, 1)^2, \tag{56}$$

with the boundary condition $u = 0$ on $\partial\Omega$.

In the first computation, the level one grid consists of two unit right triangles cutting from the unit square by a forward slash. The high level grids are the half-size refinements of the previous grid. The first three levels of grids are plotted in Figure 2. The error and the order of convergence for the both methods are shown in Tables 1 and 2. Here on triangular grids, we let $j = k + 1$ defined in (7) for computing the weak gradient $\nabla_w v$. The numerical results confirm the convergence theory.

In the next computation, we use a family of polygonal grids (with 12-side polygons) shown in Figure 3. We let the polynomial degree $j = k + 2$ for the weak

TABLE 3. Error profiles and convergence rates for (56) on polygonal grids shown in Figure 3.

level	$\|u_h - Q_0 u\|$	rate	$\|\|\|u_h - u\|\|\|$	rate	dim
	by P_1 elements with strongly enforced boundary condition				
6	0.2913E-03	2.00	0.5402E-01	1.00	15100
7	0.7289E-04	2.00	0.2701E-01	1.00	60924
8	0.1823E-04	2.00	0.1351E-01	1.00	244732
	by P_1 elements with weakly enforced boundary condition				
6	0.2982E-03	2.03	0.5333E-01	0.98	15360
7	0.7374E-04	2.02	0.2684E-01	0.99	61440
8	0.1833E-04	2.01	0.1346E-01	1.00	245760
	by P_2 elements with strongly enforced boundary condition				
6	0.1055E-05	3.00	0.7604E-03	2.00	30204
7	0.1318E-06	3.00	0.1901E-03	2.00	121852
8	0.1648E-07	3.00	0.4753E-04	2.00	489468
	by P_2 elements with weakly enforced boundary condition				
6	0.1057E-05	3.01	0.7574E-03	1.99	30720
7	0.1320E-06	3.00	0.1897E-03	2.00	122880
8	0.1649E-07	3.00	0.4748E-04	2.00	491520
	by P_3 elements with strongly enforced boundary condition				
4	0.2706E-05	3.99	0.5478E-03	2.99	3004
5	0.1696E-06	4.00	0.6862E-04	3.00	12412
6	0.1060E-07	4.00	0.8582E-05	3.00	50428
	by P_3 elements with weakly enforced boundary condition				
4	0.2813E-05	4.04	0.5421E-03	2.97	3200
5	0.1728E-06	4.02	0.6827E-04	2.99	12800
6	0.1070E-07	4.01	0.8561E-05	3.00	51200

gradient on such polygonal meshes. The rate of convergence is listed in Tables 3-4. The convergence history confirms the theory.

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References

[1] D. Arnold, F. Brezzi, B. Cockburn and D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, SIAM J. Numer. Anal., 39 (2002), 1749-1779.
[2] I. Babuška, The finite element method with penalty, Math. Comp., 27 (1973), 221-228.
[3] S. Brenner, L. Owens and L. Sung, A weakly over-penalized symmetric interior penalty method, Ele. Trans. Numer. Anal., 30 (2008), 107-127.
[4] B. Cockburn, J. Gopalakrishnan, and R. Lazarov, Unified hybridization of discontinuous Galerkin, mixed, and conforming Galerkin methods for second order elliptic problems, SIAM J. Numer. Anal., 47 (2009), 1319-136.
[5] B. Cockburn and C. Shu, The local discontinuous Galerkin finite element method for convection-diffusion systems, SIAM J. Numer. Anal., 35 (1998), 2440-2463.

TABLE 4. Error profiles and convergence rates for (56) on polygonal grids shown in Figure 3.

level	$\|u_h - Q_0 u\|$	rate	$\|\|\|u_h - u\|\|\|$	rate	dim
	by P_4 elements with strongly enforced boundary condition				
2	0.7295E-04	3.68	0.4484E-02	2.85	232
3	0.2322E-05	4.97	0.2830E-03	3.99	1068
4	0.7291E-07	4.99	0.1773E-04	4.00	4540
	by P_4 elements with weakly enforced boundary condition				
2	0.7529E-04	3.74	0.4413E-02	2.83	300
3	0.2358E-05	5.00	0.2806E-03	3.97	1200
4	0.7348E-07	5.00	0.1765E-04	3.99	4800
	by P_5 elements with strongly enforced boundary condition				
2	0.7161E-05	6.38	0.5901E-03	5.31	336
3	0.1141E-06	5.97	0.1863E-04	4.99	1516
4	0.1807E-08	5.98	0.5836E-06	5.00	6396
	by P_5 elements with weakly enforced boundary condition				
2	0.7233E-05	6.42	0.5875E-03	5.31	420
3	0.1144E-06	5.98	0.1859E-04	4.98	1680
4	0.1808E-08	5.98	0.5831E-06	5.00	6720

[6] J. Douglas Jr. and T. Dupont, Interior penalty procedures for elliptic and parabolic Galerkin methods, Computing Methods in Applied Sciences, (1976), 207-216.

[7] K. Lipnikov, G. Manzini, F. Brezzi and A. Buffa, The mimetic finite difference method for the 3D magnetostatic field problems on polyhedral meshes, J. Comput. Phys., 230 (2011), 305-328.

[8] J. Liu, S. Tavener, Z. Wang, Lowest-order weak Galerkin finite element method for Darcy flow on convex polygonal meshes, SIAM J. Sci. Comput., 40 (2018), 1229-1252.

[9] L. Mu, J. Wang, and X. Ye, weak Galerkin finite element method for second-order elliptic problems on polytopal meshes, International Journal of Numerical Analysis and Modeling, 12 (2015), 31-53.

[10] L. Mu, X. Wang and X. Ye, A modified weak Galerkin finite element method for the Stokes equations, J. Comput. Appl. Math., 275 (2015), 79-90.

[11] L. Mu, J. Wang, Y. Wang and X. Ye, A weak Galerkin mixed finite element method for biharmonic equations, Numerical Solution of Partial Differential Equations: Theory, Algorithms, and Their Applications, 45 (2013), 247-277.

[12] L. Mu, J. Wang, and X. Ye, A weak Galerkin finite element method for biharmonic equations on polytopal meshes, Numerical Methods for Partial Differential Equations, 30 (2014), 1003-1029.

[13] D. Pietro and A. Ern, Hybrid high-order methods for variable-diffusion problems on general meshes, Comptes Rendus Mathmatique, 353 (2015), 31-34.

[14] W. Reed and T. Hill. Triangular mesh methods for the neutron transport equation. Technical Report LA-UR-73-0479, Los Alamos Scientific Laboratory, Los Alamos, NM, 1973.

[15] J. Wang and X. Ye, A weak Galerkin finite element method for second-order elliptic problems. J. Comput. Appl. Math. 241 (2013), 103-115.

[16] J. Wang and X. Ye, A Weak Galerkin mixed finite element method for second-order elliptic problems, Math. Comp., 83 (2014), 2101-2126.

[17] X. Wang, N. Malluwawadu, F Gao and T. McMillan, A modified weak Galerkin finite element method, J. Comput. Appl. Math., 217 (2014), 319-327.

[18] M. Wheeler, An elliptic collocation-finite element method with interior penalties. SIAM J. Numer. Anal., 15 (1978), 152-161.

[19] X. Ye and S. Zhang, A conforming discontinuous Galerkin finite element method, International Journal of Numerical Analysis and Modeling, accepted.

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