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A FINITE DIFFERENCE SCHEME FOR CAPUTO-FABRIZIO FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this work, we consider a new fractional derivative with nonsingular kernel introduced by Caputo–Fabrizio (CF) and propose a finite difference method for computing the CF fractional derivatives. Based on an iterative technique, we can reduce the computational complexity from $O(J^2N)$ to $O(JN)$, and the corresponding storage will be cut down from $O(JN)$ to $O(N)$, which makes the computation much more efficient. Besides, by adopting piece-wise Lagrange polynomials of degrees 1, 2, and 3, we derive the second, third, and fourth order discretization formulas respectively. The error analysis and numerical experiments are carefully provided for the validation of the accuracy and efficiency of the presented method.

Key words. Caputo–Fabrizio derivative, fractional differential equations, higher order scheme.

1. Introduction

As an intensively developing area of the calculus during the past decades, fractional calculus has received much attention from both physicists and mathematicians, because it can describe the memory and hereditary properties of various materials and can faithfully capture the dynamics of physical processes in many research fields, including physics, engineering, chemistry, biology, and economics.

In the literature, there are two mostly used definitions for fractional differentiation, namely, the Riemann–Liouville (RL) and Caputo fractional operators. The RL definition plays an important role in the theory of fractional calculus and has many applications in pure mathematics, such as the definitions of new functions, see [32]. Besides, it has been found that the RL derivative is useful to characterize anomalous diffusion, Lévy flights and traps, and so forth [24]. On the other hand, practical considerations require proper definitions of fractional derivatives, which provide initial conditions with clear physical interpretation for the differential equations of fractional order. Therefore, even though the definition of Caputo derivative is more restrictive than the RL, it seems to be more welcome as well as crucial in practical applications.

Caputo's fractional derivative also has numerous applications in different areas of science [7, 9, 12, 21]. Let us start with the definition of the traditional Caputo fractional derivative [23, 32]. Given $b > 0$, $u \in H^1(0, b)$, and $0 < \alpha < 1$ with α being the fractional order, then the well-known Caputo fractional derivative of order α is defined by

$$ {}^{C}D^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s)\mathrm{d}s, \qquad t > 0. \tag{1}$$

Despite of the great success in applications, the singularity of the Caputo derivative in its kernel brings both theoretical and numerical difficulties. There are many

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investigations have been done for numerically solving differential equations with the Caputo derivative, see [3, 14, 29, 30]. A recent progress is made by J. Zhang and his collaborators on the fast algorithms for the Caputo derivatives [22, 33], where the sum-of-exponential approximation is used to approximate the smooth kernel and then an iterative scheme is introduced. For numerical strategies to overcome the initial singularity, one can refer to the recent work [26] and for stability analysis, see the work [27, 28].

To have a smooth kernel and at the same time to keep the nonlocal property, Caputo and Fabrizio proposed a new kind fractional derivative in [10]. Indeed, replacing the kernel $(t-s)^{-\alpha}$ by the function $e^{-\alpha(t-s)/(1-\alpha)}$, and replacing $\Gamma(1-\alpha)$ by $(1-\alpha)$, we obtain the new Caputo–Fabrizio (CF) fractional derivative of order $0<\alpha<1$.

Definition 1 (Caputo–Fabrizio fractional derivative)**.** *Let $0<\alpha<1$, the Caputo–Fabrizio fractional derivative of order α of a function u is defined by*

$$ {}^{\mathrm{CF}}_{0}D^{\alpha}_{t}u(t) := \frac{1}{1-\alpha}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}u'(s)\mathrm{d}s, \qquad t\geq 0. \tag{2}$$

Notice that in the original definition [10], there is a normalization factor $M(\alpha)$ in the CF derivative, which satisfies $M(0)=M(1)=1$. In a later paper [11], this factor $M(\alpha)$ is chosen to be the identity.

According to the new definition, it is clear that if u is a constant function, then ${}^{\mathrm{CF}}_{0}D^{\alpha}_{t}u=0$ as in the usual Caputo derivative. The main difference between the old and the new definitions is that, contrary to the old definition, the new kernel has no singularity at $t=s$. This suggests that the CF fractional model can describe the behavior of classical viscoelastic materials, thermal media, electromagnetic systems, etc. In fact, the original Caputo definition of fractional derivative appears to be particularly convenient for those mechanical phenomena, related with plasticity, fatigue, damage and with electromagnetic hysteresis. When these effects are not present, it seems more appropriate to use the new fractional derivative [10]. The CF derivative brings more and more attention in physics and engineering science, see [2, 4, 5, 6, 11, 17]. It is worthy to mention that there are some other kinds of nonlocal operators have been developed and used in variant time-nonlocal evolution models for describing anomalous diffusive dynamics; see for example [1, 13, 16, 35].

The objective of the present work is to develop a finite difference algorithm for the equations involving the CF fractional derivatives, which is crucial for many important applications. For example, consider the following fractional diffusion equation

$$ {}^{\mathrm{CF}}_{0}D^{\alpha}_{t}u = u_{xx} + f. \tag{3}$$

A popular method for solving such an equation is to use the piecewise linear interpolation of $u(x,t)$ on each time interval, and the order of accuracy of the method is 2. A similar method for the equation with traditional Caputo derivative has an accuracy of order $2-\alpha$, and the loss of order is due to the singularity appearing in the Caputo derivative. Moreover, the existing schemes for solving Eq. (3) require the storage of the solution at all previous time steps, so the computational complexity of these schemes is $O(J^2N)$, and the storage is $O(JN)$ on average, with J being the total number of time steps and N the number of grid points in space.