Dynamics of Lump Solutions, Rogue Wave Solutions and Traveling Wave Solutions for a (3 + 1)-Dimensional VC-BKP Equation

Ding Guo, Shou-Fu Tian*, Xiu-Bin Wang and Tian-Tian Zhang*

School of Mathematics, China University of Mining and Technology, Xuzhou 221116, PR. China.

Received 31 March 2019; Accepted (in revised version) 4 June 2019.

Abstract. The (3 + 1)-dimensional variable-coefficient B-type Kadomtsev-Petviashvili equation is studied by using the Hirota bilinear method and the graphical representations of the solutions. Breather, lump and rogue wave solutions are obtained and the influence of the parameter choice is analysed. Dynamical behavior of periodic solutions is visually shown in different planes. The rogue waves are determined and localised in time by a long wave limit of a breather with indefinitely large periods. In three dimensions the breathers have different dynamics in different planes. The traveling wave solutions are constructed by the Bäcklund transformation. The traveling wave method is used in construction of exact bright-dark soliton solutions represented by hyperbolic secant and tangent functions. The corresponding 3D figures show various properties of the solutions. The results can be used to demonstrate the interactions of shallow water waves and the ship traffic on the surface.

AMS subject classifications: 35Q51, 35Q53, 35C99, 68W30, 74J35

Key words: Breather wave solutions, rogue wave solutions, lump solutions, traveling wave solutions, bright and dark soliton solutions.

1. Introduction

Nonlinear evolution equations (NLEEs) are involved in complex physical phenomena and are used to model various problems in fluid mechanics, plasma physics, optical fibers and solid state physics [1, 2, 6, 24, 28]. In past decades such equations have been studied by mathematicians and physicists. The NLEEs demonstrate both inelastic interactions and admit localised coherent structures [5, 20, 40, 43]. The nonlinear B-type Kadomtsev-Petviashvili (KP) equation is an important representative of such equations — cf. [15, 22]. It has a variety of analytic solutions, the most crucial of which are the rogue, lump and bright-dark soliton solutions [9, 18, 29].

*Corresponding author. Email address: Ding Guo, shoufu@cumt.edu.cn, xiuwin@cumt.edu.cn, tianfeitian@cumt.edu.cn (T. Tian), ttzhang@cumt.edu.cn (T. Zhang)
The \((2 + 1)\)-dimensional KP-type equation has the form
\[
(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0,
\]
and is used to describe the shallow water waves with the weak influence of surface tension and viscosity [14].

The \((3 + 1)\)-dimensional KP-type equation
\[
u_{tx} + au_{xxx} + \beta(u_x u_y)_x + \gamma u_{xx} = 0
\]
can be viewed as a shifted \((2 + 1)\)-dimensional BKP equation for \(z = x\). If \(\delta \neq 0\), the BKP equation can be written as a \((3 + 1)\)-dimensional KP-type equation — viz.
\[
u_{tx} + au_{xxx} + \beta(u_x u_y)_x + \gamma u_{xx} + \delta u_{zz} = 0.
\]
Among other phenomena, this equation describes the propagation with a uniform speed of shallow water waves of small amplitudes in water canals of a constant depth [8, 12, 16, 17, 21, 25, 30].

Here we focus on the \((3 + 1)\)-dimensional variable-coefficient B-type Kadomtsev-Petviashvili (vc-BKP) equation
\[
(u_x + u_y + u_z)_t + au_{xxx} + \beta(u_x u_y)_x + \gamma u_{xx} + \delta u_{zz} = 0, \tag{1.1}
\]
where \(u\) is a complex function of variables \(x, y, z, t\) and \(\alpha, \beta, \gamma, \delta\) are real constants. This equation can be used to describe propagation of long waves. It also finds various applications in water percolation. If \(\alpha = 1, \beta = \chi, \gamma = \delta = -1\), the Eq. (1.1) can be reduced to the BKP equation
\[
(u_x + u_y + u_z)_t + au_{xxx} + \chi(u_x u_y)_x - (u_{xx} + u_{zz}) = 0. \tag{1.2}
\]
For a specific choice of parameters, the conservation laws and the multiple-soliton solutions of the Eq. (1.2) have been studied in [3, 42]. The BKP equation can be also used to model water waves with weakly non-linear restoring forces and frequency dispersion. Nevertheless, to the best of our knowledge, the Bäcklund transformation of the Eq. (1.1), the corresponding traveling wave solutions, breather, rogue and lump waves and bright-dark solutions have not been yet studied and we are going to address these problems here.

The rest of the paper is arranged as follows. Section 2 deals with the bilinear representation and breather wave solutions. In Section 3, lump and rogue wave solutions are derived by symbolic computation [4, 7, 10, 11, 19, 23, 26, 27, 31–39, 41, 44, 45]. The Bäcklund transformation considered in Section 4, is used to determine traveling soliton solutions. Section 5 is devoted to bright-dark soliton solutions. Our conclusions are presented in the last section.

2. Bilinear Formalism and Breather Wave Solutions

2.1. Bilinear formalism

We start with a bilinear formalism for (1.1) established by the Hirota bilinear method and the long wave limit method [13].
Theorem 2.1. The Eq. (1.1) admits the bilinear representation

\[(D_t D_x + D_y D_t + D_z D_t + \alpha D_x^3 D_y + \gamma D_x^2 D_y^2 + \delta D_x^2) f \cdot f = 0,\]

with the variable transformation

\[u = \frac{6\alpha}{\beta}(\log f)_x,\]

where \(f\) is a real function of variables \(x, y, z\) and \(t\).

Proof: Consider the potential transformation

\[u = c(t)q_x, \tag{2.1}\]

where \(c = c(t)\) is an undetermined function with respect to variable \(t\). Substituting (2.1) into the Eq. (1.1) and integrating the result with respect to \(x\), we obtain

\[c_i(q_x + q_y + q_x)_t + \alpha c q_{xxx} + \beta c^2 q_{xx} q_{xy} + \gamma c q_{xx} + \delta c q_{zz} = 0.\]

To find the bilinear form of the Eq. (1.1), the terms \(\alpha c q_{xxx}\) and \(\beta c^2 q_{xx} q_{xy}\) are associated with the known \(P\)-polynomials by

\[\alpha c q_{xxx} + \beta c^2 q_{xx} q_{xy} = \alpha c(q_{xxx} + (\alpha c / \beta) q_{xx} q_{xy}) \iff c \alpha(q_{xxx} + 3q_{xx} q_{xy}) = \alpha c P_{xxx}.\]

Thus \(c\) has to be fixed as a constant — i.e. \(c = 3\alpha / \beta\) and the Eq. (1.1) is transformed to the following Bell polynomials form:

\[E(q) = (q_x + q_y + q_z)_t + \alpha(c q_{xxx} + 3q_{xx} q_{xy}) + \gamma q_{xx} + \delta q_{zz} = 0.\]

Using the transformation \(q = 2\log(f)\), which is equivalent to \(u = (6\alpha / \beta)(\log f)\), we arrive at the equation

\[E(q) = P_{tx} + P_{yt} + P_{zt} + \alpha P_{3xy} + \gamma P_{2x} + \delta P_{2z} = 0.\]

Consequently, it follows that

\[(D_t D_x + D_y D_t + D_z D_t + \alpha D_x^3 D_y + \gamma D_x^2 D_y^2 + \delta D_x^2) f \cdot f = 0, \tag{2.2}\]

which completes the proof. \(\square\)

The \(D\)-operator is now defined by

\[D_x^{m} D_y^{n} D_t^{l}(f \cdot g) = (\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^m (\frac{\partial}{\partial y} - \frac{\partial}{\partial y'})^n (\frac{\partial}{\partial t} - \frac{\partial}{\partial t'})^l f(x, y, t) \cdot g(x', y', t') \big|_{x=x', y=y', t=t'}, \tag{2.3}\]

and so that

\[D_x D_t f \cdot f = 2(f_{xt} - f_{xt}),\]
\[D_x^2 f \cdot f = 2(f_{xx} - f_{xx}'),\]
\[D_x D_y f \cdot f = 2(f_{xy} - f_{xy}'),\]
\[D_x^2 D_y f \cdot f = 2(f_{xxy} - f_{xxy}' - f_{xxx} f_{xy} + f_{xxx} f_{xy} - f_{xxx} f_{xy}).\]
The bilinear equation (2.2) is equivalent to the form

\[ \begin{align*}
&f_{xt}f - f_x f_t + f_{yt} f - f_y f_t + f_{zz} f - f_z f_t \\
&+ \alpha (f_{3xy} f - 3 f_{2xy} f_x + 3 f_{xy} f_{xx} - f_{3xy} f_y) \\
&+ \gamma (f_{xx} f - f_y^2) + \delta (f_{zz} f - f_y^2) = 0.
\end{align*} \]

2.2. Breather wave solution

The soliton solutions \( u \) can be obtained by the bilinear transform method, if the function \( f \) is sought in the form

\[ f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2}, \] (2.4)

where

\[ \eta_i = k_i (x + p_i y + q_i z + \omega_i t) + \eta_i^0, \quad i = 1, 2, \]

\[ \omega_i = \frac{\alpha k_i^2 p_i - \delta q_i^2 - \gamma}{1 + p_i + q_i}, \quad i = 1, 2, \]

\[ A_{12} = \frac{p_1 p_2 (k_1 - k_2) (k_1 p_1 - k_2 p_2) + (k_1 q_2 - p_2 q_1)^2 - (p_1 - p_2)^2}{p_1 p_2 (k_1 + k_2) (k_1 p_1 + k_2 p_2) + (k_1 q_2 - p_2 q_1)^2 - (p_1 - p_2)^2}, \]

and \( k_i, p_i, q_i \) and \( \eta_i^0 \) are real parameters. Previously, a family of analytical solutions was determined by a suitable choice of parameters — e.g. in the Davey-Stewartson (DS) systems they are

\[ p_1 = b_1, \quad p_2 = b_1, \quad q_1 = a + ik, \quad q_2 = a - ik, \quad k_1 = ia_1, \quad k_2 = -ia_1, \quad \eta_2^0 = \eta_1^0. \]

For simplicity, we set

\[ p_1 = p_2 = 2, \quad q_1 = q_2 = 1 + 2i, \quad k_1 = k_2^* = i, \quad \eta_2^0 = \eta_1^0 = 0. \]

In this case function \( f \) takes the form

\[ \begin{align*}
f &= 2 \cosh \left( \frac{\gamma + 2 \alpha - 11 \delta}{10} t + 2z \right) \cos \left( \frac{\gamma + 2 \alpha - \delta}{5} t - x - 2\gamma - z \right) \\
&\quad - 2 \sinh \left( \frac{\gamma + 2 \alpha - 11 \delta}{10} t + 2z \right) \cos \left( \frac{\gamma + 2 \alpha - \delta}{5} t - x - 2\gamma - z \right) \\
&\quad + \frac{3}{2} \cos \left( \frac{\gamma + 2 \alpha - 11 \delta}{5} t + 4z \right) - \frac{3}{2} \sinh \left( \frac{\gamma + 2 \alpha - 11 \delta}{5} t + 4z \right) + 1, \quad (2.5)
\end{align*} \]

with real constants \( \alpha, \gamma, \delta \).

As is shown in Fig. 1, the breather wave solution has qualitatively different behavior in different planes, but the amplitudes of the excited states are bounded and are almost the same in different spaces. For \((x, t)\)-plane the corresponding solutions are displayed in Fig. 2. It can be seen that periodic line waves arise from the constant background and the amplitudes of the excited state are bounded. Besides, these periodic soliton solutions have different periods.
3. Lump Solutions and Rogue Wave Solutions

In order to generate rogue wave solutions, the long wave limit of function (2.4) has to be determined. Choosing in the Eq. (2.4) the parameters

\[ k_1 = l_1 \epsilon, \quad k_2 = l_2 \epsilon, \quad \eta_1 = \eta_2 = i\pi, \]

we have

\[ f = (\theta_1 \theta_2 + \theta_0) l_1 l_2 \epsilon^2 + O(\epsilon^3), \quad \epsilon \to 0, \]

where

\[ \theta_0 = \frac{2p_1 p_2 (p_1 - p_2)}{(p_1 q_1 - p_2 q_1)^2 - (p_1 - p_2)^2}, \]

\[ \theta_1 = \frac{- (\delta q_1^2 + \gamma) t + (p_1^2 + p_1 q_1 + p_1) y + (q_1^2 + p_1 q_1 + q_1) z + (p_1 + q_1 + 1) x}{1 + p_1 + q_1}, \]

\[ \delta = 1, \gamma = 4, \alpha = 6, k_1 = k_1^* = i, p_1 = p_2 = 2, q_1 = q_1^* = 1 + 2i. \]
the equations during their propagation. The solution $u$ of the line solitons and line rogue waves, which maintain a perfect profile without any decay during their propagation. The solution $u$ of (3.1) has the highest amplitude at $t = 0$, but it approaches a constant background for $|t| \geq 0$. The solution has a rational growing mode. Obviously, those solutions are localised in time and are the line rogue waves in previous works. On the other hand, the line rogue wave solutions can be derived with a special

\[
\theta_2 = \frac{-(\delta q_2^2 + \gamma) t + (p_2^2 + p_2 q_2 + p_2) y + (q_2^2 + p_2 q_2 + q_2) z + (p_2 + q_2 + 1) x}{1 + p_2 + q_2}
\]

Due to the gauge freedom of $f$, the rational solution can be represented in the form

\[
u = \frac{6\alpha (1 + p_1 + q_1) \theta_2 + (p_2 + q_2 + 1) \theta_1}{\delta \theta_2 + \theta_0}.
\]

Setting

\[
p_2 = p_1^*, \quad q_1^* = q_2
\]

yields $\theta_2 = \theta_1^*$ and $\theta_0 > 0$, so that the (3.1) is nonsingular. Moreover, as will be seen later on, this solution has different dynamical behavior in each plane. These two behaviors are given in the plane as an example. We set $p_1 = a_1 + i b_1, q_1 = a_2 + i b_2$, where $a_1, a_2, b_1, b_2$ are real constants.

**Case 1. Lump solutions**

If $a_1 \neq 0$, then $u$ is a constant along the trajectory $[x(t), y(t)]$, where $x$ and $y$ satisfy the equations

\[
x + a_1 y + a_2 z + \left[\delta (b_2^2 - a_2^2) - \gamma \right] \frac{(1 + a_1 + a_2) - 2a_2 b_2 (b_1 + b_2) \delta}{(1 + a_1 + a_2)^2 + (b_1 + b_2)^2} t = 0,
\]

\[
b_1 y + b_2 z + \left[\delta (a_1^2 - b_2^2) + \gamma \right] \frac{(b_1 + b_2) - 2a_2 b_2 (1 + a_1 + a_2) \delta}{(1 + a_1 + a_2)^2 + (b_1 + b_2)^2} t = 0.
\]

Moreover, for any fixed $t$ and $z$ the function $u$ tends to 0 as $(x, y)$ tends to infinity. Thus these rational solutions are permanent lumps moving on constant backgrounds. Moreover, they are rationally localised in all directions in the space. Considering the critical points under the condition (3.2), we observe the existence of the lumps — viz. one global maximum and two global minimum of (1.1). The dynamic behavior of the lump solutions in different planes is demonstrated in Fig. 3. We observe that periodic line waves arise from a constant background. Besides, the amplitude of the excited state is bounded and almost the same in different spaces.

Fig. 4 shows that the periodic solutions have qualitatively different behavior in other planes but the amplitude of the excited state is bounded and almost the same in different spaces.

**Case 2. Rogue wave solutions**

If $b_1 = 0$ — i.e. if $p_1$ is real, the solutions of (3.1) are line rogue waves in the plane, whose amplitude changes with time — cf. Fig. 5. There is an obvious difference between the line solitons and line rogue waves, which maintain a perfect profile without any decay during their propagation. The solution $u$ of (3.1) has the highest amplitude at $t = 0$, but it approaches a constant background for $|t| \geq 0$. The solution has a rational growing mode. Obviously, those solutions are localised in time and are the line rogue waves in previous works. On the other hand, the line rogue wave solutions can be derived with a special
parameter choice — e.g. if \( b_2 = 0 \), then the corresponding solutions are line rogues in the plane. These fundamental line rogue waves are demonstrated in Fig. 6, where we set \( p_1 = p_2 = 3 \) and \( q_1 = q_2^* = 1 + 2i \).

It should be pointed out, that if amplitude remains constant but the positions changes, the lump solutions can be also obtained in the planes under the same parameters. The lump phenomena appear in different planes, but the rogue waves can also appear if appropriate parameters are chosen.

The graph of the solution \( u \) in Fig. 5 shows that the periodic line waves arise from the constant background. Besides, the amplitude of the excited state is bounded and almost the same in different spaces. Considering Fig. 6, we note that the periodic solutions have qualitatively different behavior in different planes but the amplitude of the excited state is bounded and almost the same in different spaces.
4. Bäcklund Transformation and Traveling Wave Solutions

In this section, we construct the bilinear Bäcklund transformation and use it to determine traveling soliton solutions.

4.1. Bilinear Bäcklund transformation

In order to construct the Bäcklund transformation, we assume that there is another operator $D$ satisfying (2.3). Then, we have the following bilinear form

$$
(D_t D_x + D_y D_t + D_z D_t + \alpha D_x^3 D_y + \gamma D_x^2 + \delta D_z^2) f' \cdot f'' = 0.
$$

Considering the equation

$$
M = \left[\left( D_t D_x + D_y D_t + D_z D_t + \alpha D_x^3 D_y + \gamma D_x^2 + \delta D_z^2 \right) f' \cdot f'' \right] f \cdot f
- \left[\left( D_t D_x + D_y D_t + D_z D_t + \alpha D_x^3 D_y + \gamma D_x^2 + \delta D_z^2 \right) f' \cdot f \right] f' \cdot f'',
$$

Figure 5: Rogue wave solution. $\delta = 1, \gamma = 4, \alpha = 6, k_1 = k_2 = i, p_1 = p_2 = 3, q_1 = q_2 = 1 + 2i$. (a) Perspective view of the real part of the wave ($t = -6, z = 0$). (b) Perspective view of the real part of the wave ($t = 0, z = 0$). (c) Perspective view of the real part of the wave ($t = 0, z = 0$).

Figure 6: Rogue wave solution. $\delta = 1, \gamma = 4, \alpha = 6, k_1 = k_2 = i, p_1 = p_2 = 3, q_1 = q_2 = 1 + 2i$. (a) Perspective view of the real part of the wave ($y = 0, z = 0$). (b) Perspective view of the real part of the wave ($x = 0, z = 0$). (c) Perspective view of the real part of the wave ($x = 0, y = 0$).
and setting $M = 0$, we obtain

\[
\left[(D_t D_x + D_y D_t + D_z D_t + \alpha D^2_x D_y + \gamma D^2_x + \delta D^2_z) f' \cdot f' \right] f \cdot f = \left[(D_t D_x + D_y D_t + D_z D_t + \alpha D^2_x D_y + \gamma D^2_x + \delta D^2_z) f \cdot f \right] f' \cdot f'.
\]

Thus the function $f'$ can be used to find the operator $D$ in (2.3). The change of the dependent variables $f$ to $f'$ and vice versa in (4.1) does not violate the condition $M = 0$ but leads to the following equations

\[
(D_t D_j f' f') f f - (D_t D_j f) f' f' = 2D_j(D_t f' f) f f',
\]

(4.2)

and

\[
2 \left( D^2_x D_j f' f' \right) f f - 2 \left( D^2_x D_j f f \right) f' f' = D_j \left[ \left( 3D^2_x D_j f' f f + f f' \right) \left( D_j f f' \right) + (6D_i D_j f f) \left( D_j f f' \right) \right] + D_j \left[ \left( D^2_x f' f f + \left( D^2_x f' \right) D_j f f' \right) \right],
\]

(4.3)

where we used the notation $i$ and $j$ for variables $x, y, z$ and $t$. Doing the calculations

\[
2M = 2 \left( \left( D_t D_x f' f' \right) f f - \left( D_t D_x f f \right) f' f' \right) + 2 \left( \left( D_t D_y f' f' \right) f f - \left( D_t D_y f f \right) f' f' \right) + 2 \left( \left( D_t D_z f' f' \right) f f - \left( D_t D_z f f \right) f' f' \right) + 2 \left( \left( D^2_x D_z f' f f + f f' \right) \left( D_z f f' \right) + (6D_i D_z f f) \left( D_z f f' \right) \right) + D_j \left[ \left( D^2_x f' f f + \left( D^2_x f' \right) D_j f f' \right) \right].
\]

(4.4)

and taking into account (4.2)-(4.4), we derive the Bäcklund transformations of (1.1) — viz.

\[
B_1 f' f = (3D^2_x D_y + a_1 D_y + a_2 + 4\gamma D_x + a_3 D_y + a_9 D_1 - 4\delta a_{10} D_z) f' f = 0,
\]
Dynamics of Lump Solutions, Rogue Wave Solutions and Traveling Wave Solutions

4.2. Traveling wave solution

Substituting the solution \( f = 1 \) into the Eq. (1.1) yields

\[
D^n g f = D^n g = \frac{\partial^n}{\partial s^n} g, \quad n \geq 1,
\]

which is reduced to the initial variable \( u \). The Bäcklund transformation (4.5) can be split into the following group of linear equations:

\[
\begin{align*}
3f'_{xx} + a_1 f'_{y} + a_2 f' + 4\gamma f''_{y} + a_8 f''_{y} - 4\delta a_{10} f''_{z} &= 0, \\
3f'_{xx} + a_3 f'_{y} + a_4 f' &= 0, \\
f'_{xy} + a_5 f'_{x} &= 0, \\
4f'_{t} + a_6 f'_{xx} - a a_1 f'_{y}' + a a_6 f' &= 0, \\
3f'_{xx} + a a_7 f'_{x} - a_4 f' - 4a_8 f'_{x} &= 0, \\
a a_9 f'_{x} + \delta a_{11} f'_{z} &= 0, \\
f'_{t} + \delta f'_{z} + \delta a_{10} f'_{z} + \delta a_{11} f'_{t} &= 0.
\end{align*}
\]

The function \( f' \) is sought in the form

\[
f' = 1 + \eta e^{\phi_1 x + \phi_2 y + \phi_3 t - \phi_4 t}, \quad \phi_1 \neq 0,
\]

where \( \phi_1, \phi_2, \phi_3 \) and \( \phi_4 \) are constants and \( \eta \) is a real parameter. Setting \( a_2 = a_4 = a_6 = 0 \), we obtain

\[
\begin{align*}
\phi_4 &= -a_1 a \phi_1 + a \phi_1^3, \quad \phi_3 = -a_2 a \phi_1, \quad a_3 = -3\phi_1^2, \quad a_5 = -\eta^2, \\
a_7 &= 16\alpha \left( \phi_1^3 \phi_2 + \gamma \phi_1^2 + \delta \phi_2^2 - \phi_3 \phi_4 + \delta a_{11} \phi_3 \phi_4 \right) + 3\alpha \phi_1^2 \phi_2 + 4\delta \phi_3 \phi_4 - 16\phi_1 \phi_4, \\
a_8 &= 4\alpha \left( a_{11} \delta \phi_3 \phi_4 + \gamma \phi_1^2 - \phi_1^3 \phi_2 + \delta \phi_2^2 - \phi_3 \phi_4 \right) + 3\alpha \phi_1^2 \phi_2 + 4\delta \phi_3 \phi_4 - 4\phi_2 \phi_4, \\
a_{10} &= \frac{\delta a_{11} \phi_4 - \delta \phi_3 + \phi_4}{\delta \phi_1}.
\end{align*}
\]
It follows that we have the exponential wave solution

\[ u = \frac{6\alpha}{\beta} \left[ \ln f' \right]_x, \]

where

\[ f' = 1 + \eta e^{\phi_1 x + \phi_2 y - (a_9 \alpha \phi_1 / \delta) z - ( - a_1 \alpha \phi_1 + \alpha \phi_3 t ) / 4 t}. \]

Representing \( f' \) as

\[ f' = \phi_1 x + \phi_2 y + \phi_3 z - \phi_4 t, \tag{4.8} \]

substituting (4.8) into (4.6) and choosing \( a_i = 0, 2 \leq i \leq 7 \), we obtain a rational solution — viz.

\[ u = \frac{6\alpha}{\beta} \frac{\psi_1}{\psi_1 x + \psi_2 y + \psi_3 z - \psi_4 t}, \tag{4.9} \]

the graph of which is shown in Fig. 7.

Let us briefly analyse the effects of the free parameters on the amplitude and the width of the traveling wave solution. The solution of \( u \) is presented in Fig. 7. Note that the amplitude, velocity and the widths of the traveling wave stay the same during the propagation. It shows that the amplitude of the excited state is bounded and almost the same in different spaces.

5. Bright and Dark Soliton Solutions

In this section, we consider one-soliton solutions of the Eq. (1.1) by using the traveling wave method with tanh—sech functions. The influence of the parameters \( \alpha, \beta, \gamma, \delta \) on the field profile of one-soliton solutions is graphically illustrated in figures below.
5.1. The bright soliton solutions

For the bright soliton solution of the Eq. (1.1) the following theorem holds.

**Theorem 5.1.** The Eq. (1.1) has a bright soliton solution of the form
\[
U = -\frac{5m\alpha}{\beta} \sech^2 \left( mx + ny + qz - t \frac{\gamma m^2 + \delta q^2 + 4am^3 n}{m + n + q} \right),
\]  
(5.1)
where \( \alpha \beta < 0 \) and \( m, n, q \) denote the inverse widths of the soliton.

**Proof.** Consider the function
\[
U(x, y, z, t) = \lambda \cdot \sech^p (mx + ny + qz - vt),
\]  
(5.2)
where \( m, n, q \) are the inverse widths of the soliton, whereas \( \lambda \) and \( v \) are, respectively, amplitude and velocity of the soliton. The exponent \( p \) will be determined later on. Substituting (5.2) into (1.1) and equating the highest exponents at \( p + 4 \) and \( 2p + 2 \) yield \( p + 4 = 2p + 2 \) or \( p = 2 \). Setting \( \sech^{2p+2} = \sech^{2p+4} = 0 \) implies
\[
\beta \lambda^2 m^2 n^2 p^2 (2p + 2) + a \lambda m^3 n p (p + 1)(p + 2)(p + 3) = 0.
\]
It follows that
\[
\lambda = -\frac{5m\alpha}{\beta},
\]  
(5.3)
with the additional condition \( \alpha \beta < 0 \), which is needed for the solution existence. We next set the coefficient at \( \sech^p (mx + ny + qz - vt) \) to zero, so that
\[
-p^2 m \lambda - p^2 n v \lambda - p^2 q v \lambda + \gamma p^2 m^2 \lambda + \delta p^2 q^2 \lambda + \alpha p^4 m^3 n \lambda = 0,
\]
or
\[
v = \frac{\gamma m^2 + \delta q^2 + 4am^3 n}{m + n + q},
\]  
(5.4)
where \( \alpha, \gamma, \delta, \beta \) are real constants. Finally, substituting (5.3) and (5.4) into (5.2), we obtain the bright soliton solution of the \((3 + 1)\)-dimensional variable-coefficient KP equation.

The corresponding solution \( U \) is shown in Fig. 8. The amplitude, velocity and the width of the bright-soliton are not changed during the propagation but the excited state is bounded and almost the same in different spaces.

5.2. The dark soliton solutions

The dark soliton solution of the Eq. (1.1) are described by the following theorem.

**Theorem 5.2.** The Eq. (1.1) has the bright soliton solution of the form
\[
U = \frac{5m\alpha}{\beta} \tanh^2 \left( mx + ny + qz - t \frac{\gamma m^2 + \delta q^2 - 8am^3 n}{m + n + q} \right),
\]  
(5.5)
where \( \alpha \beta > 0 \) and \( m, n, q \) are the inverse widths of the soliton.
Proof. Consider the function

\[ U(x, y, z, t) = \lambda \cdot \tanh^{p}(mx + ny + qz - vt), \]  

where \( m, n, q \) are the inverse widths of the soliton and \( \lambda \) and \( v \) amplitude and velocity of the soliton, respectively. The exponent \( p \) will be determined later on. Substituting (5.6) into (1.1) and equating the highest exponents of \( p + 4 \) and \( 2p + 2 \) yield \( p + 4 = 2p + 2 \) or \( p = 2 \). Setting the coefficients at \( \tanh^{2p+2} \) and \( \tanh^{p+4} \) to zero, and collecting the coefficients at \( \tanh^p \) lead to the equations

\[
\begin{align*}
\alpha \lambda m^3 n p (p + 1)(p + 2) &- 2 \beta \lambda^2 m^2 n p^2 (p + 1) = 0, \\
2p^2 mv \lambda + 2p^2 n v \lambda + 2p^2 q v \lambda - 2\gamma p^2 m^2 \lambda - 2\delta p^2 q^2 \lambda + 4\alpha p^4 m^3 n \lambda &= 0.
\end{align*}
\]

It follows

\[
\lambda = \frac{5ma}{\beta}, \quad v = \frac{\gamma m^2 + \delta q^2 - 8am^3 n}{m + n + q},
\]

with the additional condition \( \alpha \beta > 0 \) needed for the solution existence. Finally, the dark soliton solution of the \((3 + 1)\)-dimensional variable-coefficient KP equation when these \( \lambda \) and \( v \) are substituted into the Eq. (5.6).

The results above show that the existence conditions for bright-dark soliton solutions are adverse to each other. The solution of \( U \) is displayed in Fig. 9. The amplitude, velocity and width of the bright-soliton remain the same during the propagation but the excited state is bounded and almost the same in different spaces.
6. Conclusions

We study the $(3 + 1)$-dimensional variable-coefficient B-type Kadomtsev-Petviashvili equation by using the Hirota bilinear method and the graphical representations of the solutions. The breather, lump and rogue wave solutions are obtained and the influence of the parameter choice is analysed. Dynamical behavior of the periodic solutions is visually shown in different planes. The rogue waves are obtained and localised in time by the long wave limit of breather with indefinitely large periods. In three dimensions the breathers have different dynamics in different planes. The traveling wave solutions are constructed by the Bäcklund transformation. The traveling wave method is used in construction of bright-dark soliton solutions represented by sech and tanh functions. The corresponding 3D figures show various properties of the solutions. These results can be used to demonstrate the interactions of water waves in shallow water the ship traffic on the surface.

Acknowledgments

The authors would like to thank the editor and the referees for their valuable comments and suggestions.

This work was supported by the Postgraduate Research and Practice of Educational Reform for Graduate Students in CUMT under Grant No. 2019YJSJG046, by the Natural Science Foundation of Jiangsu Province under Grant No. BK20181351, by the Six Talent Peaks Project in Jiangsu Province under Grant No. JY-059, by the Qinglan Project of Jiangsu Province of China, the National Natural Science Foundation of China under
References