A Compact Difference Scheme for Fourth-Order Temporal Multi-Term Fractional Wave Equations and Maximum Error Estimates

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Abstract. A spatial compact difference scheme for a class of fourth-order temporal multi-term fractional wave equations is developed. The original problem is reduced to a lower order system and the corresponding time fractional derivatives are approximated by the L1-formula. The unconditional stability and convergence of the difference scheme are proved by the energy method. Numerical experiments support theoretical results.

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Key words: Compact difference scheme, multi-term fractional derivatives, spatial fourth-order derivative, stability, convergence.

1. Introduction

Fractional calculus finds applications in various fields of science and technology, including physics, signal and image processing, control, mechanics, dynamic systems, biology, environmental science, materials and economics [26]. However, the non-locality and history dependence cause serious difficulties in finding solutions of fractional differential equations (FDEs). Therefore, considerable efforts have been spent on developing approximation methods for their solution. Nevertheless, the existing works mainly focus on equations with a single time fractional derivative term and some of the works on finite difference methods are reviewed here.

Murillo and Yuste [20] considered an explicit difference method for fractional diffusion-wave equations in the Caputo form, discretising time-fractional derivatives by the L2-formula. The method has first-order accuracy in time and fractional Von-Neumann approach was used in the stability analysis. Sun and Wu [29] studied a finite difference

The modelling of various phenomena leads to fractional diffusion-wave equations with the fourth-order spatial derivative. Agrawal [1] found a general solution of such equations on bounded domains. On the other hand, Guo et al. [8] studied a local discontinuous Galerkin method. Li and Wong [16] used quintic splines in spatial derivative approximation to develop a numerical method of order higher than four. They also discretised the time-fractional derivative by weighted and shifted Grünwald-Letnikov formulas [15]. Hu and Zhang [10–12] considered various finite difference schemes for fourth-order fractional diffusion-wave equations. Zhang and Pu [35] proposed a compact difference scheme for fourth-order fractional sub-diffusion systems and Vong and Wang [30] considered a compact finite difference scheme for same type of systems with the first kind of Dirichlet boundary conditions. Ji et al. [14] studied another compact difference scheme for a similar problem and Yao and Wang [33] investigated a compact difference scheme to problems with Neumann boundary conditions.

All the above mentioned works deal with fourth-order fractional diffusion-wave equations containing a single time-fractional derivative term. It is worth noting that multi-term fractional derivatives are used in visco-elastic damping [9], frequency-dependent loss and dispersion [19]. As far as numerical methods are concerned, Liu et al. [19] reduced a multi-term fractional differential equation to a system with several single-term equations, employing then a fractional predictor and corrector method. Dehghan et al. [4] applied a compact finite difference approximation in time and Galerkin spectral approximation in space to a multi-term fractional wave equation and Zhou et al. [37] studied a weak Galerkin finite element method for multi-term time-fractional diffusion equation. Using variable separation and the $L_1$-formula, Shen et al. [23] determined analytical and numerical solutions of

However, to the best of authors’ knowledge, there are only few works on approximation methods for fourth-order multi-term fractional diffusion-wave equations. Ran and Zhang [21] presented a compact difference scheme for fourth-order time-fractional sub-diffusion equations of distributed order, where a multi-term problem was resulted firstly and then an average of the spatial fourth-order derivatives was discretised together with shifted and weighted Grünwald-Letnikov formula for the approximation of multi-term time-fractional derivatives.

In the present work, we develop a numerical method to the following fourth-order temporal multi-term fractional wave equations:

\[
\begin{align*}
&D_0^\alpha D_t^\alpha u(x,t) + D_0^\alpha D_t^\alpha u(x,t) + \frac{\partial^4 u(x,t)}{\partial x^4} = f(x,t), \quad x \in (0,L), \quad t \in (0,T], \quad (1.1) \\
&u(x,0) = \phi(x), \quad \frac{\partial u(x,0)}{\partial t} = \psi(x), \quad x \in [0,L], \quad (1.2) \\
&u(0,t) = g_0(t), \quad u(L,t) = g_1(t), \quad t \in (0,T], \quad (1.3) \\
&\frac{\partial^2 u(0,t)}{\partial x^2} = \beta_0(t), \quad \frac{\partial^2 u(L,t)}{\partial x^2} = \beta_1(t), \quad t \in (0,T], \quad (1.4)
\end{align*}
\]

where \(1 < \alpha_2 \leq \alpha_1 < 2, f(x,t), \phi(x), \psi(x), g_i(t), \beta_i(t), i = 1, 2\) are given functions, and \(D_0^\alpha D_t^\alpha u(x,t)\) is the \(\alpha_i\)-th order time-fractional Caputo derivative defined by

\[
D_0^\alpha D_t^\alpha u(x,t) = \frac{1}{\Gamma(2-\alpha_i)} \int_0^t (t-s)^{1-\alpha_i} \frac{\partial^2 u(x,s)}{\partial s^2} \, ds, \quad i = 1, 2.
\]

Besides, we also assume that the compatibility conditions

\[
\begin{align*}
&\phi(0) = g_0(0), \quad \phi(L) = g_1(0), \quad \psi(0) = g'_0(0), \\
&\psi(L) = g'_1(0), \quad \phi_{xx}(0) = \beta_0(0), \quad \phi_{xx}(L) = \beta_1(0)
\end{align*}
\]

hold.

The outline of this article is as follows. Section 2 introduces preliminaries and notations. A compact difference scheme is developed in Section 3. The stability and convergence of the method in maximum norm are shown in Section 4. The results of numerical experiments are reported in Section 5 and our conclusions and remarks are drawn in Section 6.

2. Preliminaries

Let \(M\) and \(N\) be positive integers and \(x_j = jh, j = 0, 1, \ldots, M, h = L/M\) and \(t_n = n\tau, \quad n = 0, 1, \ldots, N, \tau = T/N\) be, respectively, uniform partitions of spatial and temporal intervals \([0,L]\) and \([0,T]\) . Consider the mesh function spaces \(\mathcal{U}_h = \{u|u = (u_0, u_1, \ldots, u_M)\}\).
and \( \mathcal{U}_h = \{ u | u \in \mathcal{U}_0, u_0 = u_M = 0 \} \) and the following operators acting on the space \( \mathcal{U}_h \):

\[
\delta_x u_{j-1/2} := \frac{u_j - u_{j-1}}{h}, \quad 1 \leq j \leq M,
\]

\[
\delta_x^2 u_j := \frac{1}{h} (\delta_x u_{j+1/2} - \delta_x u_{j-1/2}), \quad 1 \leq j \leq M - 1,
\]

\[
\mathcal{A} u_j := \begin{cases} 
\frac{1}{12} (u_{j-1} + 10u_j + u_{j+1}), & 1 \leq j \leq M - 1, \\
u_j, & j = 0, M.
\end{cases}
\]

On the space \( \mathcal{U}_h \) we define a discrete inner product and several norms — viz.

\[
(u, v) = h \sum_{i=1}^{M-1} u_i v_i, \quad \| u \| = \sqrt{(u, u)}, \quad \| u \|_\infty = \max_{1 \leq i \leq M-1} |u_i|,
\]

\[
\| \delta_x u \| = \sqrt{h \sum_{i=1}^{M} (\delta_x u_{i-1/2})^2}, \quad \| \delta_x^2 u \| = \sqrt{h \sum_{i=1}^{M-1} (\delta_x^2 u_i)^2}.
\]

Consider the grid function space \( \mathcal{W}_\tau = \{ w^n | 0 \leq n \leq N \} \) and for any mesh function \( w \in \mathcal{W}_\tau \) set

\[
w^{n-1/2} := \frac{w^n + w^{n-1}}{2}, \quad \delta_i w^{n-1/2} := \frac{1}{\tau} (w^n - w^{n-1}), \quad 1 \leq n \leq N,
\]

\[
\delta^a_i \left( w^{n-1/2}, q \right) := \frac{\tau^{1-a}}{\Gamma (3-a)} \left[ b_0^{(a)} \delta_i w^{n-1/2} - \sum_{k=1}^{n-1} \left( b_k^{(a)} - b_{n-k}^{(a)} \right) \delta_i w^{k-1/2} - b_{n-1}^{(a)} q \right], \quad 1 \leq n \leq N,
\]

where \( 1 < \alpha < 2, b_k^{(a)} := (k + 1)^{2-a} - k^{2-a}, k = 0, 1, \ldots. \)

Some useful lemmas are listed below.

**Lemma 2.1** (cf. Sun [27]). For any grid function \( u \in \mathcal{U}_h \), the inequalities

\[
\| u \|_\infty \leq \frac{\sqrt{T}}{2} \| \delta_x u \|, \quad \| u \| \leq \frac{L}{\sqrt{6}} \| \delta_x u \|
\]

hold.

**Lemma 2.2.** For any grid function \( u \in \mathcal{U}_h \), the inequality

\[
\| u \|_\infty \leq \frac{\sqrt{6T}}{12} L \| \delta_x^2 u \|
\]

holds.
Proof. It follows from Lemma 2.1 that
\[
\| \partial_x u \|^2 = (u, -\partial_x^2 u) \leq \|u\| \|\partial_x^2 u\| \leq \frac{L}{\sqrt{b}} \| \partial_x u \| \| \partial_x^2 u \|.
\]
Therefore,
\[
\| \partial_x u \| \leq \frac{L}{\sqrt{b}} \| \partial_x^2 u \|
\]
and using Lemma 2.1 once more, we obtain (2.1). \qed

Lemma 2.3 (cf. Liao & Sun [17]). Let
\[
x_{i+1} := x_i + h \quad \text{and} \quad \zeta(s) := (1 - s)^3 \left[ 5 - 3(1 - s)^2 \right].
\]
If \( g(x) \in C^6[x_{i-1}, x_{i+1}] \), then
\[
g''(x_{i+1}) + 10g''(x_i) + g''(x_{i-1})
= \frac{g(x_{i+1}) - 2g(x_i) + g(x_{i-1})}{h^2}
+ \frac{h^4}{360} \int_0^1 \left[ g^{(6)}(x_i - sh) + g^{(6)}(x_i + sh) \right] \zeta(s) \, ds.
\]

Lemma 2.4 (cf. Sun & Gao [28]). If \( f(t) \in C^3[t_0, t_n] \), \( 1 < \alpha < 2 \), then
\[
\frac{1}{2} \left[ c_0 D_t^\alpha f(t) \big|_{t=t_n} + c_0 D_t^\alpha f(t) \big|_{t=t_{n-1}} \right]
= \frac{\Gamma(1-\alpha)}{\Gamma(3-\alpha)} \left[ b_0^{(a)} \partial_t f^{n-1/2} - \sum_{k=1}^{n-1} \left( b_{n-k}^{(a)} - b_{n-k-1}^{(a)} \right) \partial_t f^{k-1/2} - b_{n-1}^{(a)} f'(t_0) \right] + \hat{R}^{n-1/2},
\]
where
\[
|\hat{R}^{n-1/2}| \leq \left\{ \frac{1}{6\Gamma(3-\alpha)} + \frac{1}{2\Gamma(2-\alpha)} \left[ \frac{1}{4} + \frac{\alpha - 1}{(2-\alpha)(3-\alpha)} \right] \right\} \max_{t_0 \leq t \leq t_n} |f'''(t)| t^{3-\alpha}.
\]

Lemma 2.5 (cf. Gao & Sun [6]). Let \( \{F^k|k \geq 0\} \) and \( \{G^k|k \geq 1\} \) be two non-negative sequences such that \( \{G^k\} \) is a monotone nondecreasing one and
\[
F^k \leq C \tau \sum_{l=0}^{k} F^l + G^k, \quad k = 1, 2, \ldots
\]
with a non-negative constant \( C \). If \( \tau \leq 2/(3C) \), then
\[
F^k \leq \exp \left( 3Ck\tau \right) \left( C\tau F^0 + 3G^k \right), \quad k = 1, 2, \ldots
\]
3. A Compact Difference Scheme

In this section, we develop an effective finite difference scheme for the Eqs. (1.1)-(1.4). Introducing the intermediate function \( v(x, t) = \partial^2 u(x, t) / \partial x^2 \), we rewrite the problem (1.1)-(1.4) as

\[
\begin{align*}
\frac{c_0 D_t^{a_1}}{0} u(x, t) + \frac{c_0 D_t^{a_2}}{0} u(x, t) + \frac{\partial^2 v(x, t)}{\partial x^2} &= f(x, t), \quad x \in (0, L), \quad t \in (0, T], \\
v(x, t) &= \frac{\partial^2 u(x, t)}{\partial x^2}, \quad x \in (0, L), \quad t \in [0, T],
\end{align*}
\]

(3.1)

(3.2)

Let \( u(x, 0) = \phi(x) \), \( \frac{\partial u(x, 0)}{\partial t} = \psi(x) \), \( x \in [0, L] \),

(3.3)

(3.4)

(3.5)

\( u(0, t) = g_0(t), \quad u(L, t) = g_1(t), \quad t \in (0, T] \), \( \nu(0, t) = \beta_0(t), \quad \nu(L, t) = \beta_1(t), \quad t \in (0, T] \).

Let \( U^n_i = u(x_i, t_n), V^n_i = v(x_i, t_n), \psi_i = \psi(x_i), 0 \leq i \leq M, 0 \leq n \leq N \) be grid functions. Evaluating the Eqs. (3.1) and (3.2) at the points \( (x_i, t_n) \) yields

\[
\begin{align*}
\frac{c_0 D_t^{a_1}}{0} u(x_i, t_n) + \frac{c_0 D_t^{a_2}}{0} u(x_i, t_n) + \frac{\partial^2 v(x_i, t_n)}{\partial x^2} &= f(x_i, t_n), \quad 0 \leq i \leq M, \quad 0 \leq n \leq N, \\
v(x_i, t_n) &= \frac{\partial^2 u(x_i, t_n)}{\partial x^2}, \quad 0 \leq i \leq M, \quad 0 \leq n \leq N.
\end{align*}
\]

(3.6)

(3.7)

It follows from (3.6) that

\[
\begin{align*}
&\frac{1}{2} \left[ \frac{c_0 D_t^{a_1}}{0} u(x_i, t_n) + \frac{c_0 D_t^{a_2}}{0} u(x_i, t_{n-1}) \right] + \frac{1}{2} \left[ \frac{c_0 D_t^{a_1}}{0} u(x_i, t_n) + \frac{c_0 D_t^{a_2}}{0} u(x_i, t_{n-1}) \right] \\
&\quad + \frac{1}{2} \left[ \frac{\partial^2 v(x_i, t_n)}{\partial x^2} + \frac{\partial^2 v(x_i, t_{n-1})}{\partial x^2} \right] = f_i^{n-1/2}, \quad 0 \leq i \leq M, \quad 1 \leq n \leq N,
\end{align*}
\]

(3.8)

where \( f_i^{n-1/2} = (f_i^n + f_i^{n-1}) / 2 \). Applying the operator \( \mathcal{A} \) to the Eqs. (3.8), (3.7) and using Lemmas 2.4, 2.3 and Taylor’s formula, we obtain

\[
\begin{align*}
\mathcal{A} \delta_i^{n-1/2} (U_i^{n-1/2}, \psi_i) + \mathcal{A} \delta_i^{a_2} (U_i^{n-1/2}, \psi_i) + \delta_i^2 V_i^{n-1/2} \\
= \mathcal{A} f_i^{n-1/2} + R_i^{n-1/2}, \quad 1 \leq i \leq M - 1, \quad 1 \leq n \leq N, \\
\mathcal{A} V_i^n = \delta_i^2 U_i^n + S_i^n, \quad 1 \leq i \leq M - 1, \quad 0 \leq n \leq N,
\end{align*}
\]

(3.9)

(3.10)

and note that there is a positive constant \( c_1 \) such that

\[
\begin{align*}
|R_i^{n-1/2}| &\leq c_1 \left( \tau^{3-a_1} + h^4 \right), \quad 1 \leq i \leq M - 1, \quad 1 \leq n \leq N, \\
|S_i^n| &\leq c_1 h^4, \quad 1 \leq i \leq M - 1, \quad 0 \leq n \leq N.
\end{align*}
\]

(3.11)

(3.12)

Besides, the initial boundary conditions (3.3)-(3.5) imply

\[
\begin{align*}
U_0^n = g_0(t_n), \quad U_M^n = g_1(t_n), \quad 1 \leq n \leq N, \\
V_0^n = \beta_0(t_n), \quad V_M^n = \beta_1(t_n), \quad 0 \leq n \leq N, \\
U_i^0 = \phi(x_i), \quad 0 \leq i \leq M.
\end{align*}
\]

(3.13)

(3.14)

(3.15)
Neglecting the small terms $R_i^{n-1/2}, S_i^n$ in (3.9)-(3.10), (3.13)-(3.15) and replacing the exact solution $\{U_i^n, V_i^n|0 \leq i \leq M, 0 \leq n \leq N\}$ by $\{u_i^n, v_i^n|0 \leq i \leq M, 0 \leq n \leq N\}$, we arrive at the following compact difference scheme for (3.1)-(3.5):

\[
\mathfrak{d}_t^\alpha \left( u_i^{n-1/2}, \psi_i \right) + \mathfrak{d}_t^{\alpha_2} \left( u_i^{n-1/2}, \psi_i \right) + \delta_x^2 v_i^{n-1/2} = \mathfrak{d}_f_i^{n-1/2}, \quad 1 \leq i \leq M - 1, \quad 1 \leq n \leq N,
\]

\[
\mathfrak{d}_t^\alpha \left( u_i^n, \psi_i \right) + \mathfrak{d}_t^{\alpha_2} \left( u_i^n, \psi_i \right) + \delta_x^2 u_i^n = \mathfrak{d}_f_i^n, \quad 1 \leq i \leq M - 1, \quad 0 \leq n \leq N,
\]

\[
u_0^n = g_0(t_n), \quad v_M^n = g_1(t_n), \quad 1 \leq n \leq N,
\]

\[
u_0^0 = \beta_0(t_n), \quad v_M^0 = \beta_1(t_n), \quad 0 \leq n \leq N,
\]

\[
u_0^n = \phi(x_i), \quad 0 \leq i \leq M.
\]

The unknowns $\{u_i^n|0 \leq i \leq M, 0 \leq n \leq N\}$ can be directly calculated by eliminating intermediate variables $\{v_i^n|0 \leq i \leq M, 0 \leq n \leq N\}$ — viz.

\[
\mathfrak{d}_t^2 \left( u_i^{n-1/2}, \psi_i \right) + \mathfrak{d}_t^{2\alpha_2} \left( u_i^{n-1/2}, \psi_i \right) + \delta_x^4 u_i^n = \mathfrak{d}_f_i^{n-1/2}, \quad 1 \leq i \leq M - 1, \quad 1 \leq n \leq N,
\]

\[
u_0^n = g_0(t_n), \quad u_M^n = g_1(t_n), \quad 1 \leq n \leq N,
\]

\[
u_0^n = \phi(x_i), \quad 0 \leq i \leq M,
\]

where

\[
\delta_x^4 u_i^n = \frac{1}{h^2} \left( \delta_x^2 u_{i-1}^n - 2\delta_x^2 u_i^n + \delta_x^2 u_{i+1}^n \right), \quad 1 \leq i \leq M - 1, \quad 0 \leq n \leq N,
\]

\[
\delta_x^2 u_M^n = \beta_0(t_n), \quad \delta_x^2 u_M^n = \beta_1(t_n),
\]

\[
\mathfrak{d}_t^2 u_1 = \frac{5}{6} u_1 + \frac{1}{12} u_2, \quad \mathfrak{d}_t^2 u_{M-1} = \frac{1}{12} u_{M-2} + \frac{5}{6} u_{M-1}.
\]

**4. Error of the Difference Scheme**

Consider now the stability and convergence of the difference scheme above.

**Theorem 4.1.** Let $\phi_0 = \phi_M = 0$ and let $\{u_i^n|0 \leq i \leq M, 0 \leq n \leq N\}, \{v_i^n|0 \leq i \leq M, 0 \leq n \leq N\}$ refer to the solutions of the difference scheme

\[
\mathfrak{d}_t^\alpha \left( u_i^{n-1/2}, \psi_i \right) + \mathfrak{d}_t^{\alpha_2} \left( u_i^{n-1/2}, \psi_i \right) + \delta_x^2 v_i^{n-1/2} = \mathfrak{d}_f_i^{n-1/2}, \quad 1 \leq i \leq M - 1, \quad 1 \leq n \leq N,
\]

\[
\mathfrak{d}_t^\alpha \left( u_i^n, \psi_i \right) + \mathfrak{d}_t^{\alpha_2} \left( u_i^n, \psi_i \right) + \delta_x^2 u_i^n = \mathfrak{d}_f_i^n, \quad 1 \leq i \leq M - 1, \quad 0 \leq n \leq N,
\]

\[
u_0^n = g_0(t_n), \quad u_M^n = g_1(t_n), \quad 1 \leq n \leq N,
\]

\[
u_0^0 = \beta_0(t_n), \quad v_M^0 = \beta_1(t_n), \quad 0 \leq n \leq N,
\]

\[
u_0^n = \phi(x_i), \quad 0 \leq i \leq M.
\]
If $\tau \leq 2/3$, then
\[
\|u^n\|_\infty \leq \frac{\sqrt{CL}}{12} \left[ \exp \left( \frac{3}{2} \eta \tau \right) \sqrt{\tau \|A\|} \right] \left( 1 + \|G\| \right), \quad 1 \leq n \leq N, \tag{4.6}
\]
where $G$ is defined in (4.14).

**Proof.** Set
\[
\eta_{a_1} := \tau^{a_1-1} \Gamma(3 - a_1), \quad \eta_{a_2} := \tau^{a_2-1} \Gamma(3 - a_2).
\]
It follows from (4.2) that
\[
\mathcal{A} \delta_i u_i^{n-1/2} = \delta_i \delta_x^2 u_i^{n-1/2} + \delta_i Q_i^{n-1/2}, \quad 1 \leq i \leq M - 1, \quad 1 \leq n \leq N. \tag{4.7}
\]
Computing the inner products of $\mathcal{A} \delta_i u_i^{n-1/2}$ and the Eq. (4.1) and also $\mathcal{A} v^{n-1/2}$ and the Eq. (4.7) and summing the results, we obtain
\[
\left( \frac{b^{(a_1)}}{\eta_{a_1}} + \frac{b^{(a_2)}}{\eta_{a_2}} \right) \| \mathcal{A} \delta_i u_i^{n-1/2} \|^2 + \left( \delta_x^{2} v^{n-1/2}, \mathcal{A} \delta_i u_i^{n-1/2} \right) + \left( \mathcal{A} \delta_i v^{n-1/2}, \mathcal{A} v^{n-1/2} \right)
\]
\[
= \frac{1}{\eta_{a_1}} \sum_{k=1}^{n-1} \left( b^{(a_1)}_{n-k-1} - b^{(a_1)}_{n-k} \right) \left( \mathcal{A} \delta_i u_i^{k-1/2}, \mathcal{A} \delta_i u_i^{n-1/2} \right) + b^{(a_1)}_{n-1} \left( \mathcal{A} \psi, \mathcal{A} \delta_i u_i^{n-1/2} \right)
\]
\[
+ \frac{1}{\eta_{a_2}} \sum_{k=1}^{n-1} \left( b^{(a_2)}_{n-k-1} - b^{(a_2)}_{n-k} \right) \left( \mathcal{A} \delta_i u_i^{k-1/2}, \mathcal{A} \delta_i u_i^{n-1/2} \right) + b^{(a_2)}_{n-1} \left( \mathcal{A} \psi, \mathcal{A} \delta_i u_i^{n-1/2} \right)
\]
\[
+ \left( \delta_i \delta_x^{2} u_i^{n-1/2}, \mathcal{A} v^{n-1/2} \right) + \left( p^{n-1/2}, \mathcal{A} \delta_i u_i^{n-1/2} \right) + \left( \delta_i Q_i^{n-1/2}, \mathcal{A} v^{n-1/2} \right), \quad 1 \leq n \leq N. \tag{4.8}
\]
The zero boundary conditions (4.3) and (4.4) yield
\[
\left( \delta_x^{2} v^{n-1/2}, \mathcal{A} \delta_i u_i^{n-1/2} \right) = \left( \mathcal{A} v^{n-1/2}, \delta_i \delta_x^{2} u_i^{n-1/2} \right),
\]
\[
\left( \mathcal{A} \delta_i v^{n-1/2}, \mathcal{A} v^{n-1/2} \right) = \frac{1}{2 \tau} \left( \| \mathcal{A} v^n \|^2 - \| \mathcal{A} v^{n-1} \|^2 \right).
\]
Therefore, the Eq. (4.8) takes the form
\[
\left( \frac{b^{(a_1)}}{\eta_{a_1}} + \frac{b^{(a_2)}}{\eta_{a_2}} \right) \| \mathcal{A} \delta_i u_i^{n-1/2} \|^2 + \frac{1}{2 \tau} \left( \| \mathcal{A} v^n \|^2 - \| \mathcal{A} v^{n-1} \|^2 \right)
\]
\[
= \frac{1}{\eta_{a_1}} \sum_{k=1}^{n-1} \left( b^{(a_1)}_{n-k-1} - b^{(a_1)}_{n-k} \right) \left( \mathcal{A} \delta_i u_i^{k-1/2}, \mathcal{A} \delta_i u_i^{n-1/2} \right) + b^{(a_1)}_{n-1} \left( \mathcal{A} \psi, \mathcal{A} \delta_i u_i^{n-1/2} \right)
\]
\[
+ \frac{1}{\eta_{a_2}} \sum_{k=1}^{n-1} \left( b^{(a_2)}_{n-k-1} - b^{(a_2)}_{n-k} \right) \left( \mathcal{A} \delta_i u_i^{k-1/2}, \mathcal{A} \delta_i u_i^{n-1/2} \right) + b^{(a_2)}_{n-1} \left( \mathcal{A} \psi, \mathcal{A} \delta_i u_i^{n-1/2} \right)
\]
\[
+ \left( p^{n-1/2}, \mathcal{A} \delta_i u_i^{n-1/2} \right) + \left( \delta_i Q_i^{n-1/2}, \mathcal{A} v^{n-1/2} \right), \quad 1 \leq n \leq N,
\]
and subsequent application of the Cauchy-Schwarz inequality shows that

\[
\left( \frac{b_0^{(a_1)}}{\eta_{a_1}} + \frac{b_0^{(a_2)}}{\eta_{a_2}} \right) \| \mathcal{A} \delta_t u^{n-1/2} \|^2 + \frac{1}{2\tau} \left( \| \mathcal{A} v^n \|^2 - \| \mathcal{A} v^{n-1} \|^2 \right) \\
\leq \frac{1}{\eta_{a_1}} \left[ \frac{1}{2} \sum_{k=1}^{n-1} \left( b_{n-k-1}^{(a_1)} - b_{n-k}^{(a_1)} \right) \| \mathcal{A} \delta_t u^{k-1/2} \|^2 + \frac{1}{2} \left( b_0^{(a_1)} - b_{n-1}^{(a_1)} \right) \| \mathcal{A} \delta_t u^{n-1/2} \|^2 \\
+ \frac{1}{2} b_{n-1}^{(a_1)} \left( \| \mathcal{A} \psi \|^2 + \| \mathcal{A} \delta_t u^{n-1/2} \|^2 \right) \right] \\
+ \frac{1}{\eta_{a_2}} \left[ \frac{1}{2} \sum_{k=1}^{n-1} \left( b_{n-k-1}^{(a_2)} - b_{n-k}^{(a_2)} \right) \| \mathcal{A} \delta_t u^{k-1/2} \|^2 + \frac{1}{2} \left( b_0^{(a_2)} - b_{n-1}^{(a_2)} \right) \| \mathcal{A} \delta_t u^{n-1/2} \|^2 \\
+ \frac{1}{2} b_{n-1}^{(a_2)} \left( \| \mathcal{A} \psi \|^2 + \| \mathcal{A} \delta_t u^{n-1/2} \|^2 \right) \right] \\
+ \left( p^{n-1/2}, \mathcal{A} \delta_t u^{n-1/2} \right) + \left( \delta_t Q^{n-1/2}, \mathcal{A} v^{n-1/2} \right), \quad 1 \leq n \leq N.
\]

Multiplying this inequality by 2\tau and simplifying the result, we arrive at the estimate

\[
\left( \frac{b_0^{(a_1)}}{\eta_{a_1}} + \frac{b_0^{(a_2)}}{\eta_{a_2}} \right) \| \mathcal{A} \delta_t u^{n-1/2} \|^2 + \left( \| \mathcal{A} v^n \|^2 - \| \mathcal{A} v^{n-1} \|^2 \right) \\
\leq \frac{\tau}{\eta_{a_1}} \left[ \frac{1}{2} \sum_{k=1}^{n-1} \left( b_{n-k-1}^{(a_1)} - b_{n-k}^{(a_1)} \right) \| \mathcal{A} \delta_t u^{k-1/2} \|^2 + b_{n-1}^{(a_1)} \| \mathcal{A} \psi \|^2 \right] \\
+ \frac{\tau}{\eta_{a_2}} \left[ \frac{1}{2} \sum_{k=1}^{n-1} \left( b_{n-k-1}^{(a_2)} - b_{n-k}^{(a_2)} \right) \| \mathcal{A} \delta_t u^{k-1/2} \|^2 + b_{n-1}^{(a_2)} \| \mathcal{A} \psi \|^2 \right] \\
+ 2\tau \left( p^{n-1/2}, \mathcal{A} \delta_t u^{n-1/2} \right) + 2\tau \left( \delta_t Q^{n-1/2}, \mathcal{A} v^{n-1/2} \right), \quad 1 \leq n \leq N. \quad (4.9)
\]

Setting

\[
F^n := \| \mathcal{A} v^n \|^2 + \frac{\tau}{\eta_{a_1}} \sum_{k=1}^{n} b_{n-k}^{(a_1)} \| \mathcal{A} \delta_t u^{k-1/2} \|^2 \\
+ \frac{\tau}{\eta_{a_2}} \sum_{k=1}^{n} b_{n-k}^{(a_2)} \| \mathcal{A} \delta_t u^{k-1/2} \|^2, \quad 0 \leq n \leq N, \quad (4.10)
\]

we write the inequality (4.9) in the form

\[
F^n \leq F^{n-1} + \left( \frac{b_{n-1}^{(a_1)}}{\eta_{a_1}} + \frac{b_{n-1}^{(a_2)}}{\eta_{a_2}} \right) \| \mathcal{A} \psi \|^2 + 2\tau \left( p^{n-1/2}, \mathcal{A} \delta_t u^{n-1/2} \right) \\
+ 2\tau \left( \delta_t Q^{n-1/2}, \mathcal{A} v^{n-1/2} \right), \quad 1 \leq n \leq N. \quad (4.11)
\]
Successive application of the inequality (4.11) leads to the estimate

\[ F^n \leq F^0 + \tau \sum_{k=0}^{n-1} \left( \frac{b_k^{(a_1)}}{\eta_{a_1}} + \frac{b_k^{(a_2)}}{\eta_{a_2}} \right) \| \mathcal{A} \psi \|^2 + 2\tau \sum_{k=1}^{n} \left( p^{k-1/2}, \mathcal{A} \delta_t u^{k-1/2} \right) \]

\[ + 2\tau \sum_{k=1}^{n} \left( \delta_t Q^{k-1/2}, \mathcal{A} v^{k-1/2} \right), \quad 1 \leq n \leq N, \]

and taking into account the Cauchy-Schwarz inequality, we first write

\[ F^n \leq F^0 + \tau \sum_{k=0}^{n-1} \left( \frac{b_k^{(a_1)}}{\eta_{a_1}} + \frac{b_k^{(a_2)}}{\eta_{a_2}} \right) \| \mathcal{A} \psi \|^2 \]

\[ + \tau \sum_{k=1}^{n} \left[ \frac{\eta_{a_2}}{4b_{n-k}^{(a_2)}} \| p^{k-1/2} \|^2 + \frac{b_{n-k}^{(a_2)}}{\eta_{a_2}} \| \mathcal{A} \delta_t u^{k-1/2} \|^2 \right] \]

\[ + \tau \sum_{k=1}^{n} \left[ \frac{\eta_{a_1}}{4b_{n-k}^{(a_1)}} \| p^{k-1/2} \|^2 + \frac{b_{n-k}^{(a_1)}}{\eta_{a_1}} \| \mathcal{A} \delta_t u^{k-1/2} \|^2 \right] \]

\[ + \tau \sum_{k=1}^{n} \left( \| \mathcal{A} \delta_t Q^{k-1/2} \|^2 + \| \mathcal{A} v^{k-1/2} \|^2 \right), \]

and then

\[ \| \mathcal{A}^n \psi \|^2 \leq \| \mathcal{A}^0 \psi \|^2 + \tau \sum_{k=0}^{n-1} \left( \frac{b_k^{(a_1)}}{\eta_{a_1}} + \frac{b_k^{(a_2)}}{\eta_{a_2}} \right) \| \mathcal{A} \psi \|^2 + \tau \sum_{k=1}^{n} \left( \frac{\eta_{a_1}}{4b_{n-k}^{(a_1)}} + \frac{\eta_{a_2}}{4b_{n-k}^{(a_2)}} \right) \| p^{k-1/2} \|^2 \]

\[ + \tau \sum_{k=1}^{n} \left( \| \mathcal{A} v^{k-1/2} \|^2 + \| \mathcal{A} v^{k-1/2} \|^2 \right) + \tau \sum_{k=1}^{n} \| \mathcal{A} \delta_t Q^{k-1/2} \|^2, \quad 1 \leq n \leq N. \quad (4.12) \]

The inequality

\[ \frac{\eta_{a_i}}{b_{n-k}^{(a_i)}} \leq \Gamma(2-\alpha_i) t_n^{\alpha_i-1}, \quad i = 1, 2 \]

of [28] and the relations

\[ \tau \sum_{k=0}^{n-1} \left( \frac{b_k^{(a_1)}}{\eta_{a_1}} + \frac{b_k^{(a_2)}}{\eta_{a_2}} \right) = \frac{t_n^{2-a_1}}{\Gamma(3-\alpha_1)} + \frac{t_n^{2-a_2}}{\Gamma(3-\alpha_2)} \]

show that

\[ \| \mathcal{A}^n \psi \|^2 \leq \| \mathcal{A}^0 \psi \|^2 + \frac{\tau}{2} \sum_{k=1}^{n} \left( \| \mathcal{A} v^{k-1/2} \|^2 + \| \mathcal{A} v^{k-1/2} \|^2 \right) + \left[ \frac{t_n^{2-a_1}}{\Gamma(3-\alpha_1)} + \frac{t_n^{2-a_2}}{\Gamma(3-\alpha_2)} \right] \| \mathcal{A} \psi \|^2 \]
The combination of Lemma 2.2 and inequality (4.17) produces estimate (4.6).

Let us describe the convergence in more detail.

Setting

\[ G^n := \|\mathcal{A}v^0\|^2 + \frac{1}{4} \left[ \frac{t_n^{2-\alpha_1}}{\Gamma(3-\alpha_1)} + \frac{t_n^{2-\alpha_2}}{\Gamma(3-\alpha_2)} \right] \|\mathcal{A}\psi\|^2 + \tau \sum_{k=1}^n \|\delta_k Q^{k-1/2}\|^2 \]

we write the inequality (4.13) as

\[ \|\mathcal{A}v^n\|^2 \leq \tau \sum_{k=0}^n \|\mathcal{A}v^k\|^2 + G^n, \quad 1 \leq n \leq N. \] (4.15)

It is easily seen that \( \{G^n | n \geq 1\} \) is a monotone nondecreasing sequence. Therefore, if \( \tau \leq 2/3 \), Lemma 2.5 yields

\[ \|\mathcal{A}v^n\|^2 \leq \exp(3n\tau)(\|\mathcal{A}v^0\|^2 + 3G^n), \quad 1 \leq n \leq N. \] (4.16)

Besides, it follows from (4.2) that

\[ \|\delta_x^2 u^n\| = \|\mathcal{A}v^n - Q^n\| \leq \|\mathcal{A}v^n\| + \|Q^n\| \leq \exp\left(\frac{3}{2}n\tau\right)\sqrt{\tau}\|\mathcal{A}v^0\|^2 + 3G^n + \|Q^n\|, \quad 1 \leq n \leq N. \] (4.17)

The combination of Lemma 2.2 and inequality (4.17) produces estimate (4.6).

Theorem 4.1 shows that the component \( \{u^n_i | 0 \leq i \leq M, 0 \leq n \leq N\} \) of the solution of the difference scheme (3.16)-(3.20), which is also the solution of the difference scheme (3.21)-(3.23), is unconditionally stable with respect to initial values and source term \( f(x,t) \). Let us describe the convergence in more detail.

**Theorem 4.2.** Let \( u(x,t) \in C^{(3)}_{\alpha,t}([0,L] \times [0,T]) \) and \( \{u^n_i | 0 \leq i \leq M, 0 \leq n \leq N\} \) be, respectively, solutions of the problem (1.1)-(1.4) and the difference scheme (3.21)-(3.23) and let

\[ e^n_i := u(x_i,t_n) - u^n_i, \quad 0 \leq i \leq M, \quad 0 \leq n \leq N. \]
If \( \tau \leq 2/3 \) and \( n\tau \leq T \), then

\[
\|e^n\|_\infty \leq c_2\left( \tau^{3-\alpha_1} + h^4 \right), \quad 1 \leq n \leq N, \tag{4.18}
\]

where

\[
c_2 = \frac{\sqrt{6}}{12} L^2 \left[ \exp \left( \frac{3}{2} \right) \sqrt{4c_1^2 + 3Tc_3^2 + 3[\Gamma(2-\alpha_1)T^{\alpha_1-1} + \Gamma(2-\alpha_2)T^{\alpha_2-1}]c_1^2 T/4 + c_1} \right],
\]

and the constant \( c_3 \) satisfies (4.24).

**Proof.** Let \( V^n_i = v(x_i, t_n) \) and \( \{v^n_i \mid 0 \leq i \leq M, 0 \leq n \leq N \} \) be, respectively, the components of the solutions of the problem (3.1)-(3.5) and the difference scheme (3.16)-(3.20) and let \( \zeta^n_i := V^n_i - v^n_i, 0 \leq i \leq M, 0 \leq n \leq N \). Subtracting (3.16)-(3.20) from (3.9)-(3.10) and (3.13)-(3.15), respectively, we obtain

\[
\frac{\tau^{1-\alpha_1}}{\Gamma(3-\alpha_1)} \left[ b_0^{(a_1)} \partial_t \zeta^n_i^{n-1/2} - \sum_{k=1}^{n-1} \left( b_{n-k-1}^{(a_1)} - b_{n-k}^{(a_1)} \right) \partial_t \zeta^n_i^{k-1/2} \right] + \frac{\tau^{1-\alpha_2}}{\Gamma(3-\alpha_2)} \left[ b_0^{(a_2)} \partial_t \zeta^n_i^{n-1/2} - \sum_{k=1}^{n-1} \left( b_{n-k-1}^{(a_2)} - b_{n-k}^{(a_2)} \right) \partial_t \zeta^n_i^{k-1/2} \right] + \delta_x^2 \zeta^n_i^{n-1/2} = R^n_i, \quad 1 \leq i \leq M - 1, \quad 1 \leq n \leq N, \tag{4.19}
\]

\[
\partial_t \zeta^n_i = \frac{\tau^2}{\Gamma(\alpha_1)} \delta_x^2 \zeta^n_i + S^n_i, \quad 1 \leq i \leq M - 1, \quad 0 \leq n \leq N, \tag{4.20}
\]

\[
e^n_0 = 0, \quad e^n_M = 0, \quad 1 \leq n \leq N, \tag{4.21}
\]

\[
\zeta^n_0 = 0, \quad \zeta^n_M = 0, \quad 0 \leq n \leq N, \tag{4.22}
\]

\[
e^0_i = 0, \quad 0 \leq i \leq M. \tag{4.23}
\]

The inequality (3.12) and Taylor’s formula with the integral remainder yield

\[
|\partial_t \zeta^n_i^{n-1/2}| \leq c_3 h^4, \quad 1 \leq i \leq M - 1, \quad 1 \leq n \leq N \tag{4.24}
\]

with a positive constant \( c_3 \). Using the inequalities (3.11), (3.12), (4.24) and Theorem 4.1 in (4.19)-(4.23), we obtain the estimate (4.18).

\[\square\]

5. Numerical Experiments

The applicability and computational efficiency of the method under consideration are examined via numerical experiments for equations with smooth and non-smooth solutions and homogeneous and non-homogeneous initial conditions. Let us recall that the initial weak singularity of time-fractional differential equations has been noted in [7,22]. Therefore, two experiments here are aimed to test the computational efficiency of the scheme for the problem with non-smooth solutions.

Set

\[
E(h, \tau) := \max_{0 \leq n \leq N} \|U^n - u^n\|_{\infty}.
\]
Example 5.1. We choose \( L = \pi, T = 1, \phi(x) = \psi(x) = 0, g_0(t) = g_1(t) = 0, \beta_0(t) = \beta_1(t) = 0 \) and
\[
f(x, t) = \left[ \frac{6}{\Gamma(4 - \alpha_1)} t^{3 - \alpha_1} + \frac{6}{\Gamma(4 - \alpha_2)} t^{3 - \alpha_2} + t^3 \right] \sin x
\]
in the Eqs. (1.1)-(1.4).

The exact solution of this problem is \( u(x, t) = t^3 \sin x \). Fig. 1 demonstrates the exact solution of (1.1)-(1.4) and the numerical solution obtained by the difference scheme (3.21)-(3.23) with \( h = \pi/100, \tau = 1/100 \). The graphs show that exact and numerical solutions are in excellent agreement with each other.

![Figure 1: Example 5.1. Exact and numerical solutions, \( \alpha_1 = 1.7, \alpha_2 = 1.2 \).](image)

Next we check the spatial accuracy of the difference scheme (3.21)-(3.23). For temporal step size \( \tau = 1/200,000 \), the maximum error and the numerical convergence order are shown in Table 1.

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \pi/4 )</th>
<th>( \pi/8 )</th>
<th>( \pi/16 )</th>
<th>( \pi/32 )</th>
<th>( \pi/64 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E(h, \tau) )</td>
<td>( 1.087 \times 10^{-4} )</td>
<td>( 1.120 \times 10^{-5} )</td>
<td>( 7.793 \times 10^{-6} )</td>
<td>( 1.242 \times 10^{-7} )</td>
<td>( 5.332 \times 10^{-8} )</td>
</tr>
<tr>
<td>( \log_2 \frac{E(2h, \tau)}{E(h, \tau)} )</td>
<td>—</td>
<td>4.012</td>
<td>4.001</td>
<td>3.945</td>
<td>3.842</td>
</tr>
</tbody>
</table>

Numerical results demonstrate the fourth-order convergence in space, consistent with theoretical analysis.

To determine temporal accuracy, we fix a small spatial step size \( h = \pi/500 \) to ensure that dominated error is coming from temporal approximation and find solutions for two pairs of \( \alpha_1, \alpha_2 \). Table 2 containing computational error \( E(h, \tau) \) and numerical convergence order, shows that the difference scheme (3.21)-(3.23) has \((3 - \alpha_1)\) temporal order of convergence.
Table 2: Example 5.1. Maximum error and convergence order of difference scheme (3.21)-(3.23) in time, \( h = \pi/500 \).

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>( E(h, \tau) )</th>
<th>( \log_2 \frac{E(h, 2\tau)}{E(h, \tau)} )</th>
<th>( E(h, \tau) )</th>
<th>( \log_2 \frac{E(h, 2\tau)}{E(h, \tau)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/16</td>
<td>2.010 \times 10^{-3}</td>
<td>—</td>
<td>4.299 \times 10^{-2}</td>
<td>—</td>
</tr>
<tr>
<td>1/32</td>
<td>8.015 \times 10^{-3}</td>
<td>1.326</td>
<td>1.928 \times 10^{-2}</td>
<td>1.133</td>
</tr>
<tr>
<td>1/64</td>
<td>3.207 \times 10^{-3}</td>
<td>1.322</td>
<td>8.838 \times 10^{-3}</td>
<td>1.126</td>
</tr>
<tr>
<td>1/128</td>
<td>1.287 \times 10^{-3}</td>
<td>1.317</td>
<td>4.068 \times 10^{-3}</td>
<td>1.119</td>
</tr>
<tr>
<td>1/256</td>
<td>5.182 \times 10^{-4}</td>
<td>1.313</td>
<td>1.879 \times 10^{-3}</td>
<td>1.114</td>
</tr>
</tbody>
</table>

**Example 5.2.** We choose \( L = \pi, T = 1, \phi(x) = \sin x, \psi(x) = 0, g_0(t) = g_1(t) = 0, \beta_0(t) = \beta_1(t) = 0 \) and

\[
 f(x, t) = \left[ \frac{6}{\Gamma(4 - \alpha_1)} t^{3-\alpha_1} + \frac{6}{\Gamma(4 - \alpha_2)} t^{3-\alpha_2} + t^3 + 1 \right] \sin x
\]

in the Eqs. (1.1)-(1.4).

The exact solution of this problem is \( u(x, t) = (t^3 + 1) \sin x \). Fig. 2 demonstrates the exact solution of (1.1)-(1.4) and the numerical solution obtained by the difference scheme (3.21)-(3.23) with \( h = \pi/100, \tau = 1/100 \). The solutions in Fig. 2 show a good correlation with each other.

For temporal step size \( \tau = 1/200,000 \), the maximum error and convergence rates are shown in Table 3. The fourth-order convergence in space is consistent with theoretical analysis.

To determine temporal accuracy, we fix a small spatial step size \( h = \pi/500 \) to ensure that dominated error is coming from the temporal approximation and find solutions for two pairs of \( \alpha_1, \alpha_2 \). Table 4 shows that the numerical convergence rate is about 1.3 if \( \alpha_1 = 1.7, \alpha_2 = 1.2 \) and about 1.1 if \( \alpha_1 = 1.9, \alpha_2 = 1.4 \), consistent with theoretical results.

![Exact solution](image1.png) ![Numerical solution](image2.png)

**Figure 2:** Example 5.2. Exact and approximate solutions. \( \alpha_1 = 1.7, \alpha_2 = 1.2 \).
Table 3: Example 5.2. Maximum error and convergence order of difference scheme (3.21)-(3.23) in space. \( \tau = 1/200,000, \alpha_1 = 1.7, \alpha_2 = 1.2. \)

<table>
<thead>
<tr>
<th>( h )</th>
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<th>( \pi/8 )</th>
<th>( \pi/16 )</th>
<th>( \pi/32 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E(h, \tau) )</td>
<td>( 1.271 \times 10^{-3} )</td>
<td>( 7.816 \times 10^{-5} )</td>
<td>( 4.882 \times 10^{-6} )</td>
<td>( 3.262 \times 10^{-7} )</td>
</tr>
<tr>
<td>( \log_2 \frac{E(2h, \tau)}{E(h, \tau)} )</td>
<td>—</td>
<td>4.023</td>
<td>4.001</td>
<td>3.904</td>
</tr>
</tbody>
</table>

Table 4: Example 5.2. Maximum error and convergence order of difference scheme (3.21)-(3.23) in time. \( h = \pi/500. \)

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>( \frac{(\alpha_1 = 1.7, \alpha_2 = 1.2)}{E(h, \tau)} )</th>
<th>( \log_2 \frac{E(h, 2\tau)}{E(h, \tau)} )</th>
<th>( \frac{(\alpha_1 = 1.9, \alpha_2 = 1.4)}{E(h, \tau)} )</th>
<th>( \log_2 \frac{E(h, 2\tau)}{E(h, \tau)} )</th>
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<tr>
<td>1/16</td>
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<td>1.879 \times 10^{-3}</td>
<td>1.114</td>
</tr>
</tbody>
</table>

Example 5.3. We choose \( L = \pi, T = 1, \phi(x) = \psi(x) = 0, g_0(t) = g_1(t) = 0, \beta_0(t) = \beta_1(t) = 0 \) and

\[
f(x, t) = \left[ \frac{\Gamma(\alpha_1 + \gamma + 1)}{\Gamma(\gamma + 1)} + \frac{\Gamma(\alpha_1 + \gamma + 1)}{\Gamma(\alpha_1 + \gamma - \alpha_2 + 1)} t^{\alpha_1-\alpha_2} + t^{\alpha_1} \right] t^\gamma \sin x, \quad \gamma \geq 0
\]

in the Eqs. (1.1)-(1.4).

The exact solution of this problem is \( u(x, t) = t^{\alpha_1+\gamma} \sin x. \) We determine an approximate solution by the difference scheme (3.21)-(3.23) with the parameters \( \alpha_1 = 1.2, \alpha_2 = 1.1, M = 500. \) Theoretical convergence order of this scheme in time is \( 3 - \alpha_1 = 1.8. \) Numerical results presented in Table 5 show that the convergence order is gradually polluted when \( \gamma \) decreases. The main reason for this is the presence of weak singularities in the solution at the initial time. Thus the convergence rate of the difference scheme depends on the solution regularity and can possibly be improved by using non-uniform meshes or some corrections.

The last example is aimed to test computational efficiency in the case where exact solution is unknown.

Example 5.4. We choose \( L = \pi, \phi(x) = \sin x, \psi(x) = 0, g_0(t) = g_1(t) = 0, \beta_0(t) = \beta_1(t) = 0 \) and \( f(x, t) = 0 \) in the Eqs. (1.1)-(1.4).

The convergence order of the approximate solutions is evaluated by the two-mesh principle. For sufficiently small \( h, \) the maximum of two-mesh differences is defined by

\[
E_N^0(\tau) = \max_{0 \leq i < M} \max_{0 \leq n \leq N} \left| u_i^n(h, \tau) - u_i^{2n}(h, \tau/2) \right|.
\]
Then the term

\[ \text{order}_t := \log_2 \frac{E_N^N(\tau)}{E_N^{2N}(\tau/2)} \]

numerically reflects the estimated rate of convergence in time.

Set \( T = 1 \) and \( h = \pi/500 \). Table 6 shows the numerical errors and the estimated convergence order of the scheme (3.21)-(3.23) in time. We anticipate that the regularity of the solution is different for different values of \( \alpha_1 \), better for larger \( \alpha_1 \) and the estimated

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>( \gamma = 1.8 )</th>
<th>( \gamma = 1.3 )</th>
<th>( \gamma = 0.9 )</th>
<th>( \gamma = 0.7 )</th>
<th>( \gamma = 0.5 )</th>
<th>( \gamma = 0.3 )</th>
<th>( \gamma = 0.1 )</th>
<th>( \gamma = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/16</td>
<td>( 3.283 \times 10^{-3} )</td>
<td>( 1.914 \times 10^{-3} )</td>
<td>( 6.810 \times 10^{-4} )</td>
<td>( 2.323 \times 10^{-3} )</td>
<td>( 7.254 \times 10^{-3} )</td>
<td>( 1.495 \times 10^{-4} )</td>
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<td>( 9.212 \times 10^{-4} )</td>
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<td>( 7.261 \times 10^{-3} )</td>
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<td>( 1.238 \times 10^{-5} )</td>
<td>( 7.491 \times 10^{-6} )</td>
<td>( 2.340 \times 10^{-5} )</td>
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<td>( 1.128 \times 10^{-3} )</td>
<td>( 2.004 \times 10^{-4} )</td>
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Table 5: Example 5.3. Maximum error and convergence order of difference scheme (3.21)-(3.23) in time. \( h = \pi/500 \). \( \alpha_1 = 1.2 \). \( \alpha_2 = 1.1 \).
Table 6: Example 5.4. Absolute errors and estimated convergence orders of difference scheme (3.21)-(3.23) in time, $h = \pi/500$.

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<th>order</th>
<th>$E_\infty^N(\tau)$</th>
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The convergence order is smaller than the theoretical one for smaller $\alpha_1$.

In addition, Fig. 3 shows the absolute error $E_\infty^N$ of the difference scheme (3.21)-(3.23) at different time if $M = 100$ and $N = 1000$. Note that for long time behaviour the errors are small.

6. Conclusions and Remarks

We developed a spatial compact difference scheme for a class of fourth-order temporal multi-term fractional wave equations. The original problem is reduced to a lower order system and the corresponding time fractional derivatives are approximated by the $L_1$-formula. The unconditional stability and convergence of the proposed difference scheme are proved by the energy method. Numerical experiments support theoretical results. If solution is not smooth enough, the rate of convergence deteriorates as examples show.
In the future work, the convergence rate in time is expected to be improved based on the recent work [25]. Nevertheless, the order reduction in both space and time will be proceeded simultaneously and there are new challenges to face. The last two numerical examples show that when the solution $u$ has not enough regularity, the convergence rate deteriorates, which motivates us to study this case in the near future.

Acknowledgments

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References