East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.150117.070618

Vol. **9**, No. 1, pp. 87-101
February 2019

Superconvergence of H^1-Galerkin Mixed Finite Element Methods for Elliptic Optimal Control Problems

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Received 15 January 2017; Accepted (in revised version) 07 June 2018.

Abstract. The convergence of H^1-Galerkin mixed finite element methods for elliptic optimal control problems is studied and postprocessing operators are used to establish the superconvergence for control, state and adjoint state variables. A numerical example confirms the validity of theoretical results.

AMS subject classifications: 49J20, 65N30

Key words: Elliptic equations, optimal control problems, superconvergence, H^1-Galerkin mixed finite element methods.

1. Introduction

Optimal control problems governed by partial differential equations have found wide applications in science and engineering simulations and the finite element method is one of the most powerful techniques for their solution. Various aspects of the method, including convergence and superconvergence, have been thoroughly studied — cf. [1,5,11,13,16,17, 22–26,30,31]. A systematic introduction to finite element methods for PDEs and optimal control problems is contained in [8,19].

Recently, Chen *et al.* [3,4,7,15] studied a priori error estimates and superconvergence of the Raviart-Thomas mixed finite element method for elliptic and parabolic optimal control problems. In particular, to show the superconvergence of the control, the postprocessing

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projection operator, introduced by Meyer and Rösch [22], has been used in [3,4] and the average L^2 projection operator in [7]. However, the low regularity of the control implies the convergence order $h^{3/2}$. Hou and Chen [15] discussed the superconvergence of fully discrete mixed finite element methods for parabolic optimal control problems and presented two results for the control variable derived by the use of a recovery operator and a postprocessing projection operator.

It is well-known [9] that in standard mixed finite element procedure the approximating subspaces have to satisfy the inf-sup or Ladyzhenskaya-Babuška-Brezzi (LBB) condition. This condition considerably influences the choice of suitable finite-element spaces. Therefore, non-standard mixed finite element methods for optimal control problems have been considered. Thus for elliptic optimal control problems, Guo *et al.* [12] established a priori error estimates for a splitting positive definite mixed finite element method and Hou [14] investigated a priori and a posteriori error estimates for H^1-Galerkin mixed finite element methods from [27,28]. Let us note that the last approach allows to avoid the inf-sup condition while using polynomial approximating spaces of various degree.

The main goal of this work is to study the superconvergence of H^1-Galerkin mixed finite element approximations for an elliptic control problem. In particular, we derive two approximations for the gradient of the state variable y, one of which approximates the solution $\boldsymbol{p}_h$, whereas the other is the derivative of the approximate solution y_h. To the best of the author's knowledge, these are new results in elliptic optimal control problems.

Let Ω be a bounded domain in $\mathbb{R}^2$. We consider the linear optimal control problem for state variables $\boldsymbol{p}$, y and control u with pointwise control constraint

$$\min_{u\in U_{ad}}\left\{\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{\nu}{2}\|u\|^2\right\} \tag{1.1}$$

subject to state equation

$$-\mathrm{div}(A(x)\nabla y)+cy=f+u, \quad x\in\Omega, \tag{1.2}$$

and boundary condition

$$y=0, \quad x\in\partial\Omega. \tag{1.3}$$

Let U_{ad} refer to the admissible set of the control variable — i.e.

$$U_{ad}:=\{u\in L^2(\Omega):\ a\le u\le b,\ \text{a.e. in }\Omega\},$$

where $a,b\in\mathbb{R}$ and $a<b$. We also assume that $0<c_*\le c\le c^*$, $c\in W^{1,\infty}(\Omega)$, $y_d\in H^1(\Omega)$, $\boldsymbol{p}_d\in(H^1(\Omega))^2$ and ν is a fixed positive number. Besides, let $A(x)=(a_{ij}(x))$ be a symmetric matrix-function, such that $a_{ij}(x)\in W^{1,\infty}(\Omega)$, and

$$a_*|\xi|^2\le\sum_{i,j=1}^{2}a_{ij}(x)\xi_i\xi_j\le a^*|\xi|^2 \quad \text{for all} \quad (\xi,x)\in\mathbb{R}^2\times\bar{\Omega}, \quad 0<a_*<a^*.$$

This paper is organised as follows. In Section 2, we construct an H^1-Galerkin mixed finite element approximation scheme for the optimal control problem (1.1)-(1.3) and provide