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Superconvergence of H^1-Galerkin Mixed Finite Element Methods for Elliptic Optimal Control Problems

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Abstract. The convergence of H^1-Galerkin mixed finite element methods for elliptic optimal control problems is studied and postprocessing operators are used to establish the superconvergence for control, state and adjoint state variables. A numerical example confirms the validity of theoretical results.

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1. Introduction

Optimal control problems governed by partial differential equations have found wide applications in science and engineering simulations and the finite element method is one of the most powerful techniques for their solution. Various aspects of the method, including convergence and superconvergence, have been thoroughly studied — cf. [1,5,11,13,16,17, 22–26, 30, 31]. A systematic introduction to finite element methods for PDEs and optimal control problems is contained in [8, 19].

Recently, Chen *et al.* [3,4,7,15] studied a priori error estimates and superconvergence of the Raviart-Thomas mixed finite element method for elliptic and parabolic optimal control problems. In particular, to show the superconvergence of the control, the postprocessing

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projection operator, introduced by Meyer and Rösch [22], has been used in [3,4] and the average L^2 projection operator in [7]. However, the low regularity of the control implies the convergence order $h^{3/2}$. Hou and Chen [15] discussed the superconvergence of fully discrete mixed finite element methods for parabolic optimal control problems and presented two results for the control variable derived by the use of a recovery operator and a postprocessing projection operator.

It is well-known [9] that in standard mixed finite element procedure the approximating subspaces have to satisfy the inf-sup or Ladyzhenskaya-Babuška-Brezzi (LBB) condition. This condition considerably influences the choice of suitable finite-element spaces. Therefore, non-standard mixed finite element methods for optimal control problems have been considered. Thus for elliptic optimal control problems, Guo *et al.* [12] established a priori error estimates for a splitting positive definite mixed finite element method and Hou [14] investigated a priori and a posteriori error estimates for H^1-Galerkin mixed finite element methods from [27,28]. Let us note that the last approach allows to avoid the inf-sup condition while using polynomial approximating spaces of various degree.

The main goal of this work is to study the superconvergence of H^1-Galerkin mixed finite element approximations for an elliptic control problem. In particular, we derive two approximations for the gradient of the state variable y, one of which approximates the solution $\boldsymbol{p}_h$, whereas the other is the derivative of the approximate solution y_h. To the best of the author's knowledge, these are new results in elliptic optimal control problems.

Let Ω be a bounded domain in $\mathbb{R}^2$. We consider the linear optimal control problem for state variables $\boldsymbol{p}$, y and control u with pointwise control constraint

$$\min_{u\in U_{ad}}\left\{\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{\nu}{2}\|u\|^2\right\} \tag{1.1}$$

subject to state equation

$$-\mathrm{div}(A(x)\nabla y)+cy=f+u, \quad x\in\Omega, \tag{1.2}$$

and boundary condition

$$y=0, \quad x\in\partial\Omega. \tag{1.3}$$

Let U_{ad} refer to the admissible set of the control variable — i.e.

$$U_{ad}:=\{u\in L^2(\Omega):\ a\le u\le b,\ \text{a.e. in }\Omega\},$$

where $a,b\in\mathbb{R}$ and $a<b$. We also assume that $0<c_*\le c\le c^*$, $c\in W^{1,\infty}(\Omega)$, $y_d\in H^1(\Omega)$, $\boldsymbol{p}_d\in(H^1(\Omega))^2$ and ν is a fixed positive number. Besides, let $A(x)=(a_{ij}(x))$ be a symmetric matrix-function, such that $a_{ij}(x)\in W^{1,\infty}(\Omega)$, and

$$a_*|\xi|^2\le\sum_{i,j=1}^{2}a_{ij}(x)\xi_i\xi_j\le a^*|\xi|^2 \quad \text{for all} \quad (\xi,x)\in\mathbb{R}^2\times\bar{\Omega}, \quad 0<a_*<a^*.$$

This paper is organised as follows. In Section 2, we construct an H^1-Galerkin mixed finite element approximation scheme for the optimal control problem (1.1)-(1.3) and provide equivalent optimality conditions. The main results are stated in Section 3, where superconvergence property for average L^2 projection and the approximation of control variable and also for elliptic projections and the numerical approximations of state and co-state variables are established. Applications of these results are discussed in Section 4. The numerical examples in Section 5 illustrate theoretical findings. The results obtained are summarized in Section 6.

2. Mixed Methods for Optimal Control Problems

We denote by $W^{m,p}(\Omega)$ the Sobolev spaces on Ω with the norm

$$\|v\|_{m,p}^p = \sum_{|\alpha|\le m} \|D^\alpha v\|_{L^p(\Omega)}^p$$

and the semi-norm

$$|v|_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p,$$

and let

$$W_0^{m,p}(\Omega) := \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}.$$

If $p = 2$, then we write $H^m(\Omega)$ for $W^{m,2}(\Omega)$, $H_0^m(\Omega)$ for $W_0^{m,2}(\Omega)$, $\|\cdot\|_m$ for $\|\cdot\|_{m,2}$ and $\|\cdot\|$ for $\|\cdot\|_{0,2}$. Besides, C denotes a general positive constant independent of the spatial mesh-size h used in control and state discretisation.

We start with the construction of an H^1-Galerkin mixed finite element approximation scheme for the control problem (1.1)-(1.3). For the sake of simplicity, Ω is assumed to be a convex polygon.

Consider the space $W = H_0^1(\Omega)$ and the set

$$\boldsymbol{V} = H(\mathrm{div};\Omega) = \left\{\boldsymbol{v} \in (L^2(\Omega))^2, \mathrm{div}\,\boldsymbol{v} \in L^2(\Omega)\right\}.$$

Equipped with the norm

$$\|\boldsymbol{v}\|_{\mathrm{div}} = \|\boldsymbol{v}\|_{H(\mathrm{div};\Omega)} = \left(\|\boldsymbol{v}\|_{0,\Omega}^2 + \|\mathrm{div}\,\boldsymbol{v}\|_{0,\Omega}^2\right)^{1/2},$$

and the corresponding inner product, set $\boldsymbol{V}$ becomes a Hilbert space.

Set $\boldsymbol{p} := -A\nabla y$ and introduce the mixed variational form

$$\begin{aligned}
&(c^{-1}\mathrm{div}\,\boldsymbol{p}, \mathrm{div}\,\boldsymbol{v}) + (A^{-1}\boldsymbol{p}, \boldsymbol{v}) = (c^{-1}f, \mathrm{div}\,\boldsymbol{v}) + (c^{-1}u, \mathrm{div}\,\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V},\\
&(\nabla y, \nabla w) = -(A^{-1}\boldsymbol{p}, \nabla w), \quad \forall w \in W,
\end{aligned}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$ — cf. [27]. It can be also written as

$$\begin{aligned}
&(c^{-1}\mathrm{div}\,\boldsymbol{p}, \mathrm{div}\,\boldsymbol{v}) + (A^{-1}\boldsymbol{p}, \boldsymbol{v}) = (c^{-1}f, \mathrm{div}\,\boldsymbol{v}) + (c^{-1}u, \mathrm{div}\,\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V},\\
&(A\nabla y, \nabla w) = (\mathrm{div}\,\boldsymbol{p}, w), \quad \forall w \in W.
\end{aligned}$$

Returning to the problem (1.1)-(1.3), we write it in the following weak form: Find $(\boldsymbol{p}, y, u) \in \boldsymbol{V} \times W \times U_{ad}$ such that

$$\min_{u \in U_{ad}} \left\{ \frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2 + \frac{1}{2}\|y-y_d\|^2 + \frac{\nu}{2}\|u\|^2 \right\}, \tag{2.1}$$

$$(c^{-1}\operatorname{div}\boldsymbol{p}, \operatorname{div}\boldsymbol{v}) + (A^{-1}\boldsymbol{p}, \boldsymbol{v}) = (c^{-1}f, \operatorname{div}\boldsymbol{v}) + (c^{-1}u, \operatorname{div}\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{2.2}$$

$$(A\nabla y, \nabla w) = (\operatorname{div}\boldsymbol{p}, w), \quad \forall w \in W. \tag{2.3}$$

Taking into account the convexity of the objective functional and the results of [19], we conclude that the optimal control problem (2.1)-(2.3) has a unique solution $(\boldsymbol{p}, y, u)$. Moreover, the triplet $(\boldsymbol{p}, y, u)$ is the solution of (2.1)-(2.3) if and only if there is a co-state $(\boldsymbol{q}, z) \in \boldsymbol{V} \times W$ such that $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ satisfies the optimality conditions

$$(c^{-1}\operatorname{div}\boldsymbol{p}, \operatorname{div}\boldsymbol{v}) + (A^{-1}\boldsymbol{p}, \boldsymbol{v}) = (c^{-1}f, \operatorname{div}\boldsymbol{v}) + (c^{-1}u, \operatorname{div}\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{2.4}$$

$$(A\nabla y, \nabla w) = (\operatorname{div}\boldsymbol{p}, w), \quad \forall w \in W, \tag{2.5}$$

$$(A\nabla z, \nabla w) = -(y - y_d, w), \quad \forall w \in W, \tag{2.6}$$

$$(c^{-1}\operatorname{div}\boldsymbol{q}, \operatorname{div}\boldsymbol{v}) + (A^{-1}\boldsymbol{q}, \boldsymbol{v}) = -(\boldsymbol{p}-\boldsymbol{p}_d, \boldsymbol{v}) + (z, \operatorname{div}\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{2.7}$$

$$(\nu u - c^{-1}\operatorname{div}\boldsymbol{q}, \tilde{u} - u) \geq 0, \quad \forall \tilde{u} \in U_{ad}. \tag{2.8}$$

The inequality (2.8) can be reformulated as

$$u = \max\{a, \min(b, \operatorname{div}\boldsymbol{q}/c)\}/\nu. \tag{2.9}$$

Let $\mathcal{T}_h$ be a regular rectangulation of the polygonal domain Ω, h_T the diameter of the element T ($T \in \mathcal{T}_h$) and $h := \max h_T$. We consider a finite dimensional subspace $\boldsymbol{V}_h$ of $\boldsymbol{V}$ consisting of the lowest order Raviart-Thomas mixed finite element space [9,29], namely,

$$\boldsymbol{V}_h := \{\boldsymbol{v}_h \in \boldsymbol{V} : \forall T \in \mathcal{T}_h, \boldsymbol{v}_h|_T \in Q_{1,0}(T) \times Q_{0,1}(T)\},$$

where $Q_{m,n}(T)$ denote the space of the polynomials of degree at most m and n in x and y on T, respectively. In addition, if $W_h \subset W$ is the standard linear finite element space, then the approximated space of control is defined by $U_h = U_{ad} \cap L_h$, where

$$L_h := \{l_h \in L^2(\Omega) : \forall T \in \mathcal{T}_h,\ l_h|_T = \text{constant}\}.$$

In order to introduce the relevant mixed finite element scheme, we consider three auxiliary operators.

1. The standard elliptic projection $P_h : W \to W_h$ — cf. [8], is defined by the relation

$$(A\nabla(P_h\phi - \phi), \nabla w_h) = 0, \quad \forall w_h \in W_h \tag{2.10}$$

valid for each $\phi \in W$. Note that

$$\|\phi - P_h\phi\|_s \leq Ch^{2-s}\|\phi\|_2, \quad s = 0, 1, \quad \forall \phi \in H^s(\Omega).$$

2. The Fortin projection $\Pi_h : \boldsymbol{V} \to \boldsymbol{V}_h$ — cf. [2,9], is defined by the relations

$$(\operatorname{div}(\Pi_h \boldsymbol{q} - \boldsymbol{q}), \operatorname{div} \boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h \tag{2.11}$$

valid for each $\boldsymbol{q} \in \boldsymbol{V}$. Note that

$$\begin{aligned} &\|\boldsymbol{q} - \Pi_h \boldsymbol{q}\| \le Ch\|\boldsymbol{q}\|_1, \quad \forall \boldsymbol{q} \in (H^1(\Omega))^2, \\ &\|\operatorname{div}(\boldsymbol{q} - \Pi_h \boldsymbol{q})\|_{-s} \le Ch^{1+s}\|\operatorname{div} \boldsymbol{q}\|_1, \quad s = 0,1, \quad \forall \operatorname{div} \boldsymbol{q} \in H^1(\Omega). \end{aligned} \tag{2.12}$$

3. The standard L^2-orthogonal projection $Q_h : L^2(\Omega) \to L_h$ is defined by

$$(\phi - Q_h\phi, l_h) = 0, \quad \forall l_h \in L_h \tag{2.13}$$

valid for each $\phi \in L^2(\Omega)$. Note that

$$\|\phi - Q_h\phi\|_{-s,r} \le Ch^{1+s}|\phi|_{1,r}, \quad s = 0,1, \quad \forall \phi \in W^{1,r}(\Omega).$$

The problem (2.1)-(2.3) can be now approximated by the following mixed finite element problem: Find $(\boldsymbol{p}_h, y_h, u_h) \in \boldsymbol{V}_h \times W_h \times U_h$ such that

$$\min_{u_h \in U_h} \left\{ \frac{1}{2}\|\boldsymbol{p}_h - \boldsymbol{p}_d\|^2 + \frac{1}{2}\|y_h - y_d\|^2 + \frac{\nu}{2}\|u_h\|^2 \right\}, \tag{2.14}$$

$$(c^{-1}\operatorname{div} \boldsymbol{p}_h, \operatorname{div} \boldsymbol{v}_h) + (A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) = (c^{-1}f, \operatorname{div} \boldsymbol{v}_h) + (c^{-1}u_h, \operatorname{div} \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.15}$$

$$(A\nabla y_h, \nabla w_h) = (\operatorname{div} \boldsymbol{p}_h, w_h), \quad \forall w_h \in W_h. \tag{2.16}$$

The above control problem also has a unique solution and a triplet $(\boldsymbol{p}_h, y_h, u_h)$ is the solution of (2.14)-(2.16) if and only if there is a co-state $(\boldsymbol{q}_h, z_h) \in \boldsymbol{V}_h \times W_h$ such that the terms $\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h$ satisfy the optimality conditions

$$(c^{-1}\operatorname{div} \boldsymbol{p}_h, \operatorname{div} \boldsymbol{v}_h) + (A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) = (c^{-1}f, \operatorname{div} \boldsymbol{v}_h) + (c^{-1}u_h, \operatorname{div} \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.17}$$

$$(A\nabla y_h, \nabla w_h) = (\operatorname{div} \boldsymbol{p}_h, w_h), \quad \forall w_h \in W_h, \tag{2.18}$$

$$(A\nabla z_h, \nabla w_h) = -(y_h - y_d, w_h), \quad \forall w_h \in W_h, \tag{2.19}$$

$$(c^{-1}\operatorname{div} \boldsymbol{q}_h, \operatorname{div} \boldsymbol{v}_h) + (A^{-1}\boldsymbol{q}_h, \boldsymbol{v}_h) = -(\boldsymbol{p}_h - \boldsymbol{p}_d, \boldsymbol{v}_h) + (z_h, \operatorname{div} \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.20}$$

$$(\nu u_h - c^{-1}\operatorname{div} \boldsymbol{q}_h, \tilde{u}_h - u_h) \ge 0, \quad \forall \tilde{u}_h \in U_h. \tag{2.21}$$

The control inequality (2.21) can be reformulated as

$$u_h = \max\{a, \min(b, \operatorname{div} \boldsymbol{q}_h/(Q_h c))\}/\nu.$$

Let us introduce intermediate variables needed in what follows. For a control function $\tilde{u} \in U_{ad}$, let $(\boldsymbol{p}_h(\tilde{u}), y_h(\tilde{u}), \boldsymbol{q}_h(\tilde{u}), z_h(\tilde{u})) \in (\boldsymbol{V}_h \times W_h)^2$ be the corresponding discrete state solution such that

$$(c^{-1}\operatorname{div} \boldsymbol{p}_h(\tilde{u}), \operatorname{div} \boldsymbol{v}_h) + (A^{-1}\boldsymbol{p}_h(\tilde{u}), \boldsymbol{v}_h) = (c^{-1}f, \operatorname{div} \boldsymbol{v}_h) + (c^{-1}\tilde{u}, \operatorname{div} \boldsymbol{v}_h), \tag{2.22}$$

$$(A\nabla y_h(\tilde{u}), \nabla w_h) = (\operatorname{div} \boldsymbol{p}_h(\tilde{u}), w_h), \tag{2.23}$$

$$(A\nabla z_h(\tilde{u}), \nabla w_h) = -(y_h(\tilde{u}) - y_d, w_h), \tag{2.24}$$

$$(c^{-1}\operatorname{div} \boldsymbol{q}_h(\tilde{u}), \operatorname{div} \boldsymbol{v}_h) + (A^{-1}\boldsymbol{q}_h(\tilde{u}), \boldsymbol{v}_h) = -(\boldsymbol{p}_h(\tilde{u}) - \boldsymbol{p}_d, \boldsymbol{v}_h) + (z_h(\tilde{u}), \operatorname{div} \boldsymbol{v}_h), \tag{2.25}$$

for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$.

Thus, according to above arguments, the exact and approximate solutions can be written as

$$\begin{aligned}
(\boldsymbol{p}, y, \boldsymbol{q}, z) &= (\boldsymbol{p}(u), y(u), \boldsymbol{q}(u), z(u)),\\
(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h) &= (\boldsymbol{p}_h(u_h), y_h(u_h), \boldsymbol{q}_h(u_h), z_h(u_h)).
\end{aligned}$$

3. Superconvergence Analysis

In this section, we provide a detailed superconvergence analysis for optimal control problems, starting with auxiliary results.

Lemma 3.1. *Let* $(\boldsymbol{p}, y, \boldsymbol{q}, z)$ *and* $(\boldsymbol{p}_h(u), y_h(u), \boldsymbol{q}_h(u), z_h(u))$ *be, respectively, the solutions of* (2.4)-(2.8) *and* (2.22)-(2.25) *with* $\tilde{u} = u$. *If* $\boldsymbol{p}, \boldsymbol{q} \in (H^2(\Omega))^2$ *and* $y, z \in H^2(\Omega)$, *then*

$$\begin{aligned}
&\|\Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u)\|_{\rm div} + \|\nabla(P_h y - y_h(u))\| \leq Ch^{3/2},\\
&\|\Pi_h \boldsymbol{q} - \boldsymbol{q}_h(u)\|_{\rm div} + \|\nabla(P_h z - z_h(u))\| \leq Ch^{3/2}.
\end{aligned}$$

Proof. It follows from the Eqs. (2.4)-(2.7), (2.22)-(2.25) and (2.10) that

$$\begin{aligned}
&(c^{-1}{\rm div}\,(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u)), {\rm div}\,\boldsymbol{v}_h) + (A^{-1}(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u)), \boldsymbol{v}_h)\\
&= -(c^{-1}{\rm div}\,(\boldsymbol{p} - \Pi_h \boldsymbol{p}), {\rm div}\,\boldsymbol{v}_h) - (A^{-1}(\boldsymbol{p} - \Pi_h \boldsymbol{p}), \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \qquad (3.1)\\
&(A\nabla(P_h y - y_h(u)), \nabla w_h)\\
&= ({\rm div}\,(\boldsymbol{p} - \Pi_h \boldsymbol{p}), w_h) + ({\rm div}\,(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u)), w_h), \quad \forall w_h \in W_h, \qquad (3.2)\\
&(A\nabla(P_h z - z_h(u)), \nabla w_h)\\
&= -(y - P_h y, w_h) - (P_h y - y_h(u), w_h), \quad \forall w_h \in W_h, \qquad (3.3)\\
&(c^{-1}{\rm div}\,(\Pi_h \boldsymbol{q} - \boldsymbol{q}_h(u)), {\rm div}\,\boldsymbol{v}_h) + (A^{-1}(\Pi_h \boldsymbol{q} - \boldsymbol{q}_h(u)), \boldsymbol{v}_h)\\
&= -(c^{-1}{\rm div}\,(\boldsymbol{q} - \Pi_h \boldsymbol{q}), {\rm div}\,\boldsymbol{v}_h) - (A^{-1}(\boldsymbol{q} - \Pi_h \boldsymbol{q}), \boldsymbol{v}_h) - (\boldsymbol{p} - \Pi_h \boldsymbol{p}, \boldsymbol{v}_h)\\
&\quad - (\Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u), \boldsymbol{v}_h) + (z - P_h z, {\rm div}\,\boldsymbol{v}_h) + (P_h z - z_h(u), {\rm div}\,\boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \qquad (3.4)
\end{aligned}$$

Choosing $\Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u)$ for $\boldsymbol{v}_h$ in the Eq. (3.1), we rewrite this equation as

$$\begin{aligned}
&(c^{-1}{\rm div}\,(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u)), {\rm div}\,(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u))) + (A^{-1}(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u)), \Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u))\\
&= -(c^{-1}{\rm div}\,(\boldsymbol{p} - \Pi_h \boldsymbol{p}), {\rm div}\,(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u))) - (A^{-1}(\boldsymbol{p} - \Pi_h \boldsymbol{p}), \Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u)). \qquad (3.5)
\end{aligned}$$

According to the proof of Theorems 4.1, 5.1 and Example 6.2 in [10], for any $\boldsymbol{p} \in \boldsymbol{V}$ and $\boldsymbol{v}_h \in \boldsymbol{V}_h$ the inequality

$$(A^{-1}(\boldsymbol{p} - \Pi_h \boldsymbol{p}), \boldsymbol{v}_h) \leq Ch^{3/2}\|\boldsymbol{p}\|_2(\|\boldsymbol{v}_h\| + \|{\rm div}\,\boldsymbol{v}_h\|) \qquad (3.6)$$

holds. The application of (2.11), (2.12) and the Cauchy inequality yield

$$\begin{aligned}
&(c^{-1}{\rm div}\,(\boldsymbol{p} - \Pi_h \boldsymbol{p}), {\rm div}\,(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u)))\\
&= ((c^{-1} - Q_h(c^{-1})){\rm div}\,(\boldsymbol{p} - \Pi_h \boldsymbol{p}), {\rm div}\,(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u)))\\
&\leq Ch\|{\rm div}\,(\boldsymbol{p} - \Pi_h \boldsymbol{p})\| \cdot \|c^{-1}\|_{1,\infty}\|{\rm div}\,(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u))\|\\
&\leq Ch^2\|\boldsymbol{p}\|_2\|c^{-1}\|_{1,\infty}\|{\rm div}\,(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u))\|. \qquad (3.7)
\end{aligned}$$

Moreover, it follows from (3.5)-(3.7) and the properties of A and c that

$$\|\Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u)\|_{\text{div}} \le Ch^{3/2}\|\boldsymbol{p}\|_2 \|c^{-1}\|_{0,\infty}. \tag{3.8}$$

Analogously, we choose $P_h y - y_h(u)$ for w_h in (3.2) and obtain

$$\begin{aligned}&(A\nabla(P_h y - y_h(u)), \nabla(P_h y - y_h(u)))\\ &= (\text{div}\,(\boldsymbol{p} - \Pi_h \boldsymbol{p}), P_h y - y_h(u)) + (\text{div}\,(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u)), P_h y - y_h(u)).\end{aligned} \tag{3.9}$$

Applying again the estimate (2.12) along with the Cauchy and Poincare inequalities, we see that

$$\begin{aligned}(\text{div}\,(\boldsymbol{p} - \Pi_h \boldsymbol{p}), P_h y - y_h(u)) &\le \|\text{div}\,(\boldsymbol{p} - \Pi_h \boldsymbol{p})\|_{-1}\|P_h y - y_h(u)\|_1\\ &\le Ch^2\|\boldsymbol{p}\|_2\|\nabla(P_h y - y_h(u))\|, \end{aligned} \tag{3.10}$$

$$(\text{div}\,(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u)), P_h y - y_h(u)) \le C\|\text{div}\,(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u))\| \cdot \|\nabla(P_h y - y_h(u))\|. \tag{3.11}$$

Relations (3.9)-(3.11) and the properties of A yield

$$\|\nabla(P_h y - y_h(u))\| \le C\|\text{div}\,(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u))\| + Ch^2\|\boldsymbol{p}\|_2. \tag{3.12}$$

Choosing now $P_h z - z_h(u)$ for w_h in (3.3) and using the Poincare and Cauchy inequalities, we obtain

$$\|\nabla(P_h z - z_h(u))\| \le C\|\nabla(P_h y - y_h(u))\| + Ch^2\|y\|_2. \tag{3.13}$$

Similar procedure with the term $\boldsymbol{v}_h = \Pi_h \boldsymbol{q} - \boldsymbol{q}_h(u)$ in (3.4) yields

$$\begin{aligned}\|\Pi_h \boldsymbol{q} - \boldsymbol{q}_h(u)\|_{\text{div}} &\le Ch^2(\|\boldsymbol{p}\|_2 + \|z\|_2)\|c^{-1}\|_{1,\infty} + Ch^{3/2}(\|\boldsymbol{p}\|_2 + \|\boldsymbol{q}\|_2)\\ &\quad + C(\|\Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u)\| + \|\nabla(P_h z - z_h(u))\|),\end{aligned} \tag{3.14}$$

and the combination of the inequalities (3.8), (3.12)-(3.14) completes the proof. □

Lemma 3.2. *If* $(\boldsymbol{p}_h(Q_h u), y_h(Q_h u), \boldsymbol{q}_h(Q_h u), z_h(Q_h u))$ *and* $(\boldsymbol{p}_h(u), y_h(u), \boldsymbol{q}_h(u), z_h(u))$ *are the solutions of the problem* (2.22)-(2.25) *when* $\tilde{u}$ *is, respectively, chosen as* $Q_h u$ *and* u*, then*

$$\begin{aligned}&\|\nabla(y_h(u) - y_h(Q_h u))\| + \|\boldsymbol{p}_h(u) - \boldsymbol{p}_h(Q_h u)\|_{\text{div}} \le Ch^2,\\ &\|\nabla(z_h(u) - z_h(Q_h u))\| + \|\boldsymbol{q}_h(u) - \boldsymbol{q}_h(Q_h u)\|_{\text{div}} \le Ch^2.\end{aligned}$$

Proof. Setting $\tilde{u} = Q_h u$ and $\tilde{u} = u$ in (2.22)-(2.25), we obtain

$$\begin{aligned}&(c^{-1}\text{div}\,(\boldsymbol{p}_h(u) - \boldsymbol{p}_h(Q_h u)), \text{div}\,\boldsymbol{v}_h) + (A^{-1}(\boldsymbol{p}_h(u) - \boldsymbol{p}_h(Q_h u)), \boldsymbol{v}_h)\\ &= (c^{-1}(u - Q_h u), \text{div}\,\boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,\\ &(A\nabla(y_h(u) - y_h(Q_h u)), \nabla w_h) = (\text{div}\,(\boldsymbol{p}_h(u) - \boldsymbol{p}_h(Q_h u)), w_h), \quad \forall w_h \in W_h,\\ &(A\nabla(z_h(u) - z_h(Q_h u)), \nabla w_h) = -(y_h(u) - y_h(Q_h u), w_h), \quad \forall w_h \in W_h,\\ &(c^{-1}\text{div}\,(\boldsymbol{q}_h(u) - \boldsymbol{q}_h(Q_h u)), \text{div}\,\boldsymbol{v}_h) + (A^{-1}(\boldsymbol{q}_h(u) - \boldsymbol{q}_h(Q_h u)), \boldsymbol{v}_h)\\ &= -(\boldsymbol{p}_h(u) - \boldsymbol{p}_h(Q_h u), \boldsymbol{v}_h) + (z_h(u) - z_h(Q_h u), \text{div}\,\boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.\end{aligned}$$

Since

$$(c^{-1}(u-Q_hu), \mathrm{div}\,\boldsymbol{v}_h) = ((c^{-1}-Q_h(c^{-1}))(u-Q_hu), \mathrm{div}\,\boldsymbol{v}_h)$$
$$\leq Ch^2\|u\|_1\|c^{-1}\|_{1,\infty}\|\mathrm{div}\,\boldsymbol{v}_h\|,$$

then, analogously to the proof of Lemma 3.1, one can employ the stability estimates to finish the proof. □

Lemma 3.3. *If* $(\boldsymbol{p}_h(Q_hu), y_h(Q_hu), \boldsymbol{q}_h(Q_hu), z_h(Q_hu))$ *and* $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h)$ *are the solutions of* (2.22)-(2.25) *when* $\tilde{u}$ *is, respectively, chosen as* Q_hu *and* u_h*, then*

$$(c^{-1}\mathrm{div}\,(\boldsymbol{q}_h(Q_hu)-\boldsymbol{q}_h), Q_hu-u_h) \leq 0. \tag{3.15}$$

Proof. Setting $\tilde{u}=Q_hu$ and $\tilde{u}=u_h$ in (2.22)-(2.25) we obtain

$$(c^{-1}\mathrm{div}\,(\boldsymbol{p}_h-\boldsymbol{p}_h(Q_hu)), \mathrm{div}\,\boldsymbol{v}_h) + (A^{-1}(\boldsymbol{p}_h-\boldsymbol{p}_h(Q_hu)), \boldsymbol{v}_h)$$
$$= (c^{-1}(u_h-Q_hu), \mathrm{div}\,\boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.16}$$
$$(A\nabla(y_h-y_h(Q_hu)), \nabla w_h) = (\mathrm{div}\,(\boldsymbol{p}_h-\boldsymbol{p}_h(Q_hu)), w_h), \quad \forall w_h \in W_h, \tag{3.17}$$
$$(A\nabla(z_h-z_h(Q_hu)), \nabla w_h) = -(y_h-y_h(Q_hu), w_h), \quad \forall w_h \in W_h, \tag{3.18}$$
$$(c^{-1}\mathrm{div}\,(\boldsymbol{q}_h-\boldsymbol{q}_h(Q_hu)), \mathrm{div}\,\boldsymbol{v}_h) + (A^{-1}(\boldsymbol{q}_h-\boldsymbol{q}_h(Q_hu)), \boldsymbol{v}_h)$$
$$= -(\boldsymbol{p}_h-\boldsymbol{p}_h(Q_hu), \boldsymbol{v}_h) + (z_h-z_h(Q_hu), \mathrm{div}\,\boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{3.19}$$

Replacing $\boldsymbol{v}_h$ by $\boldsymbol{q}_h(Q_hu)-\boldsymbol{q}_h$ in (3.16), w_h by $z_h(Q_hu)-z_h$ in (3.17), w_h by $-(y_h(Q_hu)-y_h)$ in (3.18) and $\boldsymbol{v}_h$ by $-(\boldsymbol{p}_h(Q_hu)-\boldsymbol{p}_h)$ in (3.19) and summing the resulting equations, we obtain the equation

$$(c^{-1}\mathrm{div}\,(\boldsymbol{q}_h(Q_hu)-\boldsymbol{q}_h), Q_hu-u_h) = (c^{-1}(Q_hu-u_h), \mathrm{div}\,(\boldsymbol{q}_h(Q_hu)-\boldsymbol{q}_h))$$
$$= -\|y_h-y_h(Q_hu)\|^2 - \|\boldsymbol{p}_h-\boldsymbol{p}_h(Q_hu)\|^2,$$

and the inequality (3.15) follows. □

We now can discuss the superconvergence property for the control variable. Considering the sets

$$\Omega^+ = \left\{\bigcup T : T \subset \Omega, a < u(x)|_T < b\right\},$$
$$\Omega^0 = \left\{\bigcup T : T \subset \Omega, u(x)|_T \equiv a \text{ or } u(x)|_T \equiv b\right\},$$
$$\Omega^- = \Omega\backslash(\Omega^+ \cup \Omega^0)$$

we observe that they do not have common points and $\Omega = \Omega^+ \cup \Omega^0 \cup \Omega^-$. We also assume that u and $\mathscr{T}_h$ are regular — i.e. $\mathrm{meas}(\Omega^-) \leq Ch$ — cf. [22].

Theorem 3.1. *Let* u *and* u_h *be, respectively, the solutions of the problems* (2.4)-(2.8) *and* (2.17)-(2.21) *and* $\mathrm{div}\,\boldsymbol{q} \in W^{1,\infty}(\Omega)$*. Then, under conditions of Lemma* 3.1*, the inequality*

$$\|Q_hu-u_h\| \leq Ch^{3/2} \tag{3.20}$$

holds.

Proof. Set $\tilde{u} = u_h$ in (2.8) and $\tilde{u}_h = Q_h u$ in (2.21). Then

$$(\nu u - c^{-1}\mathrm{div}\boldsymbol{q}, u_h - u) \geq 0,$$
$$(\nu u_h - c^{-1}\mathrm{div}\boldsymbol{q}_h, Q_h u - u_h) \geq 0.$$

Noting that $u_h - u = u_h - Q_h u + Q_h u - u$ and summing the above inequalities, we obtain

$$(\nu u_h - \nu u + c^{-1}\mathrm{div}(\boldsymbol{q} - \boldsymbol{q}_h), Q_h u - u_h) + (\nu u - c^{-1}\mathrm{div}\boldsymbol{q}, Q_h u - u) \geq 0. \tag{3.21}$$

Therefore, the relations (3.21), (2.13) yield

$$\begin{aligned} \nu\|Q_h u - u_h\|^2 &= \nu(Q_h u - u, Q_h u - u_h) + \nu(u - u_h, Q_h u - u_h) \\ &\leq (c^{-1}\mathrm{div}(\boldsymbol{q} - \boldsymbol{q}_h), Q_h u - u_h) + (\nu u - c^{-1}\mathrm{div}\boldsymbol{q}, Q_h u - u) \\ &= (c^{-1}\mathrm{div}(\boldsymbol{q} - \Pi_h\boldsymbol{q}), Q_h u - u_h) + (c^{-1}\mathrm{div}(\Pi_h\boldsymbol{q} - \boldsymbol{q}_h(u)), Q_h u - u_h) \\ &\quad + (c^{-1}\mathrm{div}(\boldsymbol{q}_h(u) - \boldsymbol{q}_h(Q_h u)), Q_h u - u_h) + (c^{-1}\mathrm{div}(\boldsymbol{q}_h(Q_h u) - \boldsymbol{q}_h), Q_h u - u_h) \\ &\quad + (\nu u - c^{-1}\mathrm{div}\boldsymbol{q}, Q_h u - u) =: \sum_{i=1}^{5} I_i. \end{aligned} \tag{3.22}$$

In order to estimate the terms I_i, $i = 1,2,3,4,5$ we will use the Cauchy inequality and some other results. Thus, considering I_1, we employ (2.11), (2.12), so that

$$I_1 = ((c^{-1} - Q_h(c^{-1}))\mathrm{div}(\boldsymbol{q} - \Pi_h\boldsymbol{q}), Q_h u - u_h) \leq Ch^4\|\boldsymbol{q}\|_2^2 + \frac{\nu}{4}\|Q_h u - u_h\|^2. \tag{3.23}$$

For I_2 and I_3, we, respectively, use Lemmas 3.1 and 3.2, thus obtaining

$$I_2 \leq C\|\mathrm{div}(\Pi_h\boldsymbol{q} - \boldsymbol{q}_h(u))\|^2 + \frac{\nu}{4}\|Q_h u - u_h\|^2 \leq Ch^3 + \frac{\nu}{4}\|Q_h u - u_h\|^2, \tag{3.24}$$

$$I_3 \leq C\|\mathrm{div}(\boldsymbol{q}_h(u) - \boldsymbol{q}_h(Q_h u))\| \cdot \|Q_h u - u_h\| \leq Ch^4 + \frac{\nu}{4}\|Q_h u - u_h\|^2. \tag{3.25}$$

For I_4, Lemma 3.3 shows that

$$I_4 \leq 0. \tag{3.26}$$

The term I_5 can be estimated analogously to the considerations in [7, Theorem 5.1]. Thus

$$I_5 = (\nu u - c^{-1}\mathrm{div}\boldsymbol{q}, Q_h u - u) \leq Ch^3(\|u\|_{1,\infty}^2 + \|c^{-1}\|_{1,\infty}^2\|\mathrm{div}\boldsymbol{q}\|_{1,\infty}^2). \tag{3.27}$$

The inequality (3.20) now follows from (3.22)-(3.27). □

For the state variables and the adjoint state variables the superconvergence property can be obtained from Theorem 3.1, analogously to considerations in Lemma 3.1.

Theorem 3.2. *If* $(y, \boldsymbol{p}, z, \boldsymbol{q})$ *and* $(y_h, \boldsymbol{p}_h, z_h, \boldsymbol{q}_h)$ *are, respectively, the solutions of the problems* (2.4)-(2.8) *and* (2.17)-(2.21)*, then under the conditions of Theorem* 3.1*, the inequalities*

$$\|\Pi_h\boldsymbol{p} - \boldsymbol{p}_h\|_{\mathrm{div}} + \|\nabla(P_h y - y_h)\| \leq Ch^{3/2},$$
$$\|\Pi_h\boldsymbol{q} - \boldsymbol{q}_h\|_{\mathrm{div}} + \|\nabla(P_h z - z_h)\| \leq Ch^{3/2}$$

hold.

4. Applications

Here we consider the applications of the above results. Let I_{2h}^2 refer to the higher order interpolation operator defined in [21]. It is known that

$$\|v - I_{2h}^2 v\|_1 \le Ch^2\|v\|_3, \quad \forall v \in H^3(\Omega), \tag{4.1}$$

$$I_{2h}^2 I_h = I_{2h}^2, \tag{4.2}$$

$$\|I_{2h}^2 v\|_1 \le C\|v\|_1, \quad \forall v \in W_h. \tag{4.3}$$

Theorem 4.1. *If $y,z \in H^3(\Omega)$, then under the conditions of Theorem 3.2, the inequalities*

$$\|y - I_{2h}^2 y_h\|_1 \le Ch^{3/2}, \tag{4.4}$$

$$\|z - I_{2h}^2 z_h\|_1 \le Ch^{3/2} \tag{4.5}$$

hold.

Proof. It follows from (4.2) and (4.3) that

$$\begin{aligned} &y - I_{2h}^2 y_h = y - I_{2h}^2 y + I_{2h}^2(I_h y - P_h y) + I_{2h}^2(P_h y - y_h), \\ &\|y - I_{2h}^2 y_h\|_1 \le \|y - I_{2h}^2 y\|_1 + C\|I_h y - P_h y\|_1 + C\|P_h y - y_h\|_1. \end{aligned} \tag{4.6}$$

Moreover, according to [21, Theorem 2.1.1], one has

$$\|I_h y - P_h y\|_1 \le Ch^2\|y\|_3, \tag{4.7}$$

and the estimate (4.4) now follows from (4.6)-(4.7), (4.1), Theorem 3.2 and the Poincare inequality. The estimate (4.5) can be proven analogously. □

In order to obtain global superconvergence for the vector-valued functions, we employ the higher order interpolation postprocessing method from [20]. Consider a large rectangular elements partition $\mathscr{T}_{2h}$, which is a coarse mesh on $\mathscr{T}_h$ — viz. each element $\tau \in \mathscr{T}_{2h}$ consists of four neighboring rectangular elements of $\mathscr{T}_h$. We denote by $\boldsymbol{V}_{2h}$ the Raviart-Thomas mixed finite element space of the order $k = 1$ — i.e.

$$\boldsymbol{V}_{2h} := \{\boldsymbol{v} \in \boldsymbol{V} : \forall \tau \in \mathscr{T}_{2h}, \boldsymbol{v}|_\tau \in Q_{2,1}(\tau) \times Q_{1,2}(\tau)\},$$

and let Π_{2h} be the corresponding Raviart-Thomas projection — cf. [9, 29]:

$$\Pi_{2h} : \boldsymbol{V} \to \boldsymbol{V}_{2h}.$$

It is known [20] that

$$\Pi_{2h}\Pi_h = \Pi_{2h} \text{ and } \|\Pi_{2h}\boldsymbol{v}_h\|_{\text{div}} \le C\|\boldsymbol{v}_h\|_{\text{div}}, \quad \text{for all} \quad \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{4.8}$$

Theorem 4.2. *If $(y, \boldsymbol{p}, z, \boldsymbol{q})$ and $(y_h, \boldsymbol{p}_h, z_h, \boldsymbol{q}_h)$ are, respectively, the solutions of problems* (2.4)-(2.8) *and* (2.17)-(2.21)*, then under the conditions of Theorem* 3.2*, the estimates*

$$\|\boldsymbol{p} - \Pi_{2h}\boldsymbol{p}_h\|_{\text{div}} \le Ch^{3/2},$$

$$\|\boldsymbol{q} - \Pi_{2h}\boldsymbol{q}_h\|_{\mathrm{div}} \le Ch^{3/2} \tag{4.9}$$

hold.

Proof. It follows from (4.8) that

$$\boldsymbol{p} - \Pi_{2h}\boldsymbol{p}_h = \boldsymbol{p} - \Pi_{2h}\boldsymbol{p} + \Pi_{2h}(\Pi_h\boldsymbol{p} - \boldsymbol{p}_h),$$

and Theorem 3.2 and (4.8) yield

$$\|\boldsymbol{p} - \Pi_{2h}\boldsymbol{p}_h\|_{\mathrm{div}} \le \|\boldsymbol{p} - \Pi_{2h}\boldsymbol{p}\|_{\mathrm{div}} + C\|\Pi_h\boldsymbol{p} - \boldsymbol{p}_h\|_{\mathrm{div}} \le Ch^{3/2}.$$

The estimate (4.9) can be obtained analogously. □

The accuracy of the control approximation on a global scale can be improved by using postprocessing methods. For this, we can employ a recovery operator G_h. Let $G_h v$ be a continuous piecewise linear function without zero boundary constraint. The values of $G_h v$ at the nodes are found by using the least-squares arguments over the element patches surrounding the nodes — cf. the definition of R_h in [18]. Another possibility is provided by the postprocessing projection operator of the discrete co-state to the admissible set [22], so that

$$\hat{u} = \max\{a, \min(b, \operatorname{div}\Pi_{2h}\boldsymbol{q}_h/c)\}/\nu. \tag{4.10}$$

The global superconvergence results for the control variable are described by the following theorems.

Theorem 4.3 (cf. Chen *et al.* [6, Theorem 3.3])**.** *Let $u \in W^{1,\infty}(\Omega)$. If u and u_h are respectively the solutions of* (2.4)-(2.8) *and* (2.17)-(2.21)*, then under the conditions of Theorem* 3.1*, the estimate*

$$\|u - G_h u_h\| \le Ch^{3/2}$$

holds.

Theorem 4.4. *If u is the solution of* (2.4)-(2.8) *and $\hat{u}$ the function constructed in* (4.10)*, then under conditions of Theorem* 4.2*, the estimate*

$$\|u - \hat{u}\| \le Ch^{3/2}$$

holds.

Proof. It follows from (2.9) and (4.10) that

$$|u - \hat{u}| \le C|\operatorname{div}(\boldsymbol{q} - \Pi_{2h}\boldsymbol{q}_h)|, \tag{4.11}$$

and taking into account (4.11) and (4.9), we finish the proof. □

5. Numerical Experiments

We want to illustrate our theoretical results by a numerical example. Following the previous considerations, we approximate the control function u by piecewise constant functions, the variables $\boldsymbol{p}$ and $\boldsymbol{q}$ by the lowest order Raviart-Thomas mixed finite element functions and the variables y and z by piecewise linear finite element functions. Besides, let A be the unit matrix, $\Omega := [0,1]\times[0,1]$ and $\nu = c = 1$.

Example 5.1. We consider the following two-dimensional elliptic optimal control problem

$$\min_{u\in U_{ad}} \left\{ \frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2 + \frac{1}{2}\|y-y_d\|^2 + \frac{1}{2}\|u-u_0\|^2 \right\} \tag{5.1}$$

subject to the state equation

$$\mathrm{div}\boldsymbol{p} + y = f + u, \quad \boldsymbol{p} = -\nabla y, \tag{5.2}$$

where

$$\begin{aligned} & y = \sin(\pi x_1)\sin(\pi x_2), \\ & \boldsymbol{p} = \boldsymbol{q} = -\begin{pmatrix} \pi\cos(\pi x_1)\sin(\pi x_2) \\ \pi\sin(\pi x_1)\cos(\pi x_2) \end{pmatrix}, \\ & z = 2\pi^2\sin(\pi x_1)\sin(\pi x_2), \\ & u_0 = 10 - 5\sin\left(\frac{\pi x_1}{2}\right) - 5\sin(\pi x_2), \\ & u = \max\{5, \min(10, u_0 + \mathrm{div}\boldsymbol{q})\}. \end{aligned} \tag{5.3}$$

The source function f and the desired states y_d and $\boldsymbol{p}_d$ are determined from the information above. Tables 1-3 show the errors $\|u-u_h\|$, $\|Q_h u - u_h\|$, $\|u - G_h u_h\|$, $\|y-y_h\|_1$, $\|z-z_h\|_1$, $\|\nabla(P_h y - y_h)\|$ and $\|\nabla(P_h z - z_h)\|$ and convergence order on a sequence of uniformly refined meshes. These estimates clearly confirm the theoretical finding of the previous sections. Let us also note the Figs. 1-3, which display the postprocessing solution $G_h u_h$ and the numerical solutions of u and y on the 64×64 mesh.

Table 1: Numerical results for the control u.

h	$\|u-u_h\|$	Rate	$\|Q_h u-u_h\|$	Rate	$\|u-G_h u_h\|$	Rate
1/16	3.8423e-1	-	7.7460e-2	-	3.3348e-1	-
1/32	1.9355e-1	0.99	2.7595e-2	1.49	1.2404e-1	1.43
1/64	9.8336e-2	0.98	6.7080e-3	2.03	4.4252e-2	1.49
1/128	4.9199e-2	1.00	3.1698e-3	1.08	1.5212e-2	1.54
1/256	2.4632e-2	1.00	1.1102e-3	1.51	5.3692e-3	1.50

Table 2: Numerical results for the state y.

h	$\|y-y_h\|_1$	Rate	$\|\nabla(P_h y - y_h)\|$	Rate
1/16	1.2690e-1	-	1.5276e-2	-
1/32	6.3063e-2	1.00	3.5411e-3	2.10
1/64	3.1494e-2	1.00	9.6151e-4	1.87
1/128	1.5741e-2	1.00	2.3464e-4	2.03
1/256	7.8698e-3	1.00	5.7143e-5	2.03

Table 3: Numerical results for the co-state z.

h	$\|z-z_h\|_1$	Rate	$\|\nabla(P_h z - z_h)\|$	Rate
1/16	2.4849	-	6.4837e-3	-
1/32	1.2427	1.00	8.4130e-4	2.94
1/64	6.2135e-1	1.00	1.2241e-4	2.78
1/128	3.1068e-1	1.00	2.1331e-5	2.52
1/256	1.5534e-1	1.00	4.5250e-6	2.23

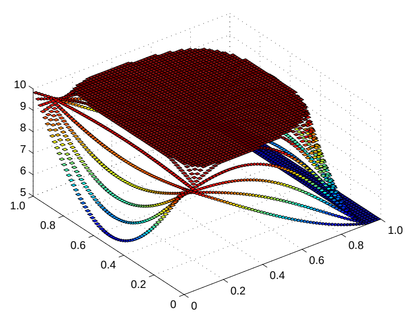

Figure 1: Approximate solution u_h, $h = 1/64$.

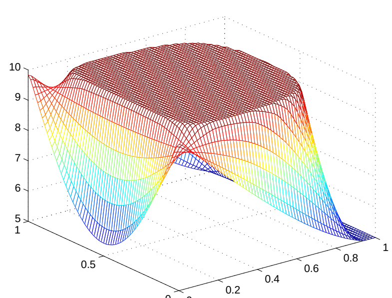

Figure 2: Continuous piecewise linear function $G_h u_h$, $h = 1/64$.

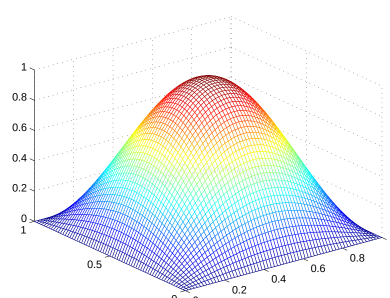

Figure 3: Approximate solution y_h, $h = 1/64$.

6. Conclusions

We proved the superconvergence of H^1-Galerkin mixed finite element methods for the linear elliptic optimal control problem (1.1)-(1.3). The approach we employ, has not been applied to this type of optimal control problems before. It can be also used to study a priori error estimates, superconvergence and a posteriori error estimates in such mixed finite element methods for parabolic optimal control problems.

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