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A Variational Iteration Method Involving Adomian Polynomials for a Strongly Nonlinear Boundary Value Problem

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Abstract. A variational iteration method involving Adomian polynomials to solve a strongly nonlinear boundary value problem is considered. After its convergence is established, the efficiency and accuracy of the proposed method are tested on problems with exponential nonlinearity.

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Key words: Boundary value problem, variational iteration method, Adomian polynomials, convergence.

1. Introduction

The nonlinear boundary value problem

$$
\begin{aligned}
&y'' + \frac{m}{x}y' + f(x,y) = 0, \quad 0 \le x \le 1, \\
&\alpha_1 y(0) + \beta_1 y'(0) = \gamma_1, \quad m = 0 \quad \text{or} \quad y'(0) = 0, \quad m > 0, \\
&\alpha_2 y(1) + \beta_2 y'(1) = \gamma_2,
\end{aligned}
\tag{1.1}
$$

where $\alpha_i, \beta_i, \gamma_i$, $i = 1,2$ are finite constants, arises in various applications, including thermal explosions [10], tumor growth models [7], electroosmotic flows [11,12,15], modelling of heat sources in human head [26], and oxygen diffusion [42]. The unique solvability of the problem (1.1) for $m \ge 1$ and boundary conditions $y'(0) = 0$ and $y(1) = B$ was established by Chawla & Shivkumar [19], while the more general case of nonlinear boundary conditions was studied by Garner & Shivaji [27]. In order to find approximate solutions of the problem, various numerical methods have been used — e.g. the Adomian decomposition method [39], the Taylor series method [18], a variational iteration method [49, 58]. The first two of these methods experience convergence difficulties, while the third one is

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restricted to the solution of the problem (1.1) with functions $f(x,y)$ belonging to a special class of non-linear polynomials. To avoid these difficulties, modifications of the last two methods have been suggested [13, 14, 16].

The variational iteration method proposed by He [30–32] and its modifications [38,56] have a high convergence rate and small error, so they are widely regarded as a good tool for solving functional equations [1,28,29,33,34,37,41,44,45,51,53,59] arising in nonlinear science and engineering problems [5, 6, 40, 50, 54, 55, 60, 61]. Based on the variational iteration algorithm I, the convergence of the method has been extensively studied in [29, 45, 51–53]. Recently, Chang [17] used the variational iteration algorithm II to prove the convergence of the method for two-point diffusion problems. For more details about the method and its applications we refer the reader to [36,37] and references therein.

The concept of Adomian polynomials was introduced by Adomian [8] in 1976. Later, Adomian and Rach [9] presented a formal formula to generate the Adomian polynomials for all form of nonlinearity. Since then, various algorithms for calculating the Adomian polynomials have been proposed to improve computational efficiency [2,22,23,47,57]. Symbolic implementation of several recurrence algorithms by using *MATHEMATICA* or *MAPLE* was also developed — cf. [20, 21, 23, 46]. The convergence of the Adomian polynomial series has been also discussed in [2, 22, 48].

Here, we combine a variational iteration method with Adomian polynomials to obtain the approximate solutions of a strongly nonlinear boundary value problem. In contrast to the above mentioned methods, our approach does not require any additional tools. The key idea is that the nonlinear terms in the correction functionals are decomposed into a series of Adomian polynomials, as this simplifies the computations considerably. Sufficient conditions for the method convergence are established, and test examples involving exponential nonlinearity, demonstrate the efficiency of the algorithm. The errors do not depend on a specific location of a point $x \in [0,1]$, and are valid for the whole domain considered. This approach can be applied to various problems involving other differential equations with a strong nonlinearity.

2. Convergence

According to He [35, 36] and Chang [17], the variational iteration algorithms I and II for the problem (1.1) have the form

$$\begin{aligned} y_{n+1}(x) &= y_n(x) + \int_0^x \lambda(s;x)\Big[y_n''(s) + \frac{m}{s}y_n'(s) + f(s,y_n(s))\Big]\mathrm{d}s, \quad n \geq 0, \\ y_{n+1}(x) &= y_0(x) + \int_0^x \lambda(s;x) f(s,y_n(s))\mathrm{d}s, \quad n \geq 0, \end{aligned} \tag{2.1}$$

where $y_0(x) := y(0) + y'(0)x$ and $\lambda(s;x)$ is the Lagrange multiplier [49, 58] defined by