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A Fourth Order Finite Difference Method for Time-Space Fractional Diffusion Equations

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Abstract. A finite difference method for a class of time-space fractional diffusion equations is considered. The trapezoidal formula and a fourth-order fractional compact difference scheme are, respectively, used in temporal and spatial discretisations and the method stability is studied. Theoretical estimates of the convergence in the L_2-norm are shown to be $\mathcal{O}(\tau^2 + h^4)$, where τ and h are time and space mesh sizes. Numerical examples confirm theoretical results.

AMS subject classifications: 65M06, 65M12, 35R11

Key words: Fractional diffusion equation, Riesz derivative, high-order approximation, stability, convergence.

1. Introduction

Fractional partial differential equations play an important role in modeling of anomalous phenomena and in complex systems. Physically significant models in anomalous diffusion deal with the limiting distribution of a group of particles in a stochastic process. These limits can be determined by using continuous time random walks, where each random particle jump occurs after a random waiting time. Fractional derivatives in space represent

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large particle jumps, while long waiting time is associated with fractional derivatives in time. Anomalous diffusion is used to interpret the hydrogen effect on the morphology of silicon electrodes in electrochemical conditions [26] and in nonlinear electrophoresis [6]. The anomalous diffusion patterns in amorphous electro-active materials can be considered by fractional calculus [7, 8]. Fractional dynamics can cause the statistics change for the joint velocity-position probability density function of single particles in turbulent flow [25]. Water transport in unsaturated soils is described by the fractional generalisation of Richardson's law [38].

To obtain analytic solutions of fractional partial differential equations [40] one can use Fourier transform, Adomian decomposition [37], Laplace transform [27], shifted Legendre polynomials [1] and Mellin transform. There is a large number of works devoted to analytic methods for fractional differential equations, but closed form solutions have been only found for a small class of equations. Therefore, various numerical methods have been developed recently, with $L1$-formula often used for the approximate the Caputo time-fractional derivative — cf. Refs. [11,13,24,33,34,41,42,46]. Although, these methods are unconditionally stable, they cannot achieve the second order time accuracy [22,36,45]. For solving time–fractional differential equations a modified $L1$-formula, called $L1$-2-formula, has been proposed in [23]. Another modification — viz. $L2$-1_σ-formula, was introduced in [2], with the corresponding difference scheme retaining the second-order temporal convergence, and a number of new high-order approximations to the Caputo time fractional derivative have been suggested in [9, 32]. They extend $L1$-2-formula and have the numerical accuracy of order $r-\alpha$, where $r \geq 4$ is a positive integer. A finite difference method for space variables and $L1$-approximation of the Caputo fractional derivative was used to find a numerical solution of the time-fractional sub-diffusion equation on an unbounded domain in two-dimensional space [35]. Tang [47] used trapezoidal rule in a finite difference method for partial integro-differential equations with a weakly singular kernel, Chen *et al.* [14] proposed fractional trapezoidal rule (FTR) difference scheme, combining the second order difference quotient for spatial discretisation and FTR alternating direction implicit method in solving a two-dimensional fractional evolution equation. In order to speed up the evaluation of the Caputo fractional derivative in $L2$-1_σ-formula, Yan *et al.* [49] employed an exponential approximation of the kernel function in the Caputo fractional derivative. Ji and Sun [31] discretised time fractional-order derivative by the second-order shifted and weighted Grünwald-Letnikov difference operator and Chen *et al.* [12] used a fractional trapezoidal rule type difference scheme with second order accuracy both in temporal and spatial directions.

Numerical methods for time-space fractional diffusion equations having a high-order of accuracy are less studied. Thus Ding *et al.* [16] considered a fourth-order scheme for space Riesz fractional diffusion equation and Ding *et al.* [17] used Fourier analysis to apply 6-th, 8-th, 10-th and 12-th order schemes to the Riesz derivative. Later on, Ding and Li [18] developed five high-order algorithms for the Riesz derivatives and applied them to the Riesz-type turbulent diffusion equation. High-order finite difference schemes for one- and two-dimensional time-space fractional sub-diffusion equations are proposed in [39]. A finite difference scheme with second-order accuracy in both time and space directions