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A Reduced Finite Element Formulation for Space Fractional Partial Differential Equation

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Abstract. A framework for solving space fractional partial differential equations by reduced finite element methods is proposed. In particular, we construct reduced bases, study their properties and use them in numerical schemes. The stability and convergence of these methods are investigated. Two numerical examples show that such an approach has a high efficiency and a low computational cost.

AMS subject classifications: 65M10, 78A48

Key words: Proper orthogonal decomposition, finite element method, space fractional partial differential equation.

1. Introduction

Recently, fractional partial differential equations (FPDEs) have found applications in various fields of science and engineering physics [28], chemistry [35], finance [29], thus becoming a hot research topic. Finding exact solutions of such equations is a challenging problem and even if an equation is solved, the solutions can have a complicated structure. Therefore, the development of approximation methods for FPDEs attracted a substantial interest — cf. Refs. [6–9, 13, 15, 19–21]. The non-locality of fractional operators is one of key issues alleviating computational load and resources, especially in the case of space FPDEs. In practical problems, both the accuracy and computational efficiency of the model play an important role [14]. The goal of this work is to develop a reduced basis method for space FPDEs, which would balance the accuracy and efficiency.

The central idea of the reduced basis approach is the identification of a suitable, problem dependent basis, efficiently representing the solutions of FPDEs from snapshots — i.e. one has to find the most representative snapshots and determine whether the corresponding basis is rich enough. One of sampling strategies consists in constructing the singular value decomposition of the large number of snapshots. More precisely, one has to derive

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the proper orthogonal decomposition (POD) — cf. Lumley [23]. Let us recall that POD was successfully applied to various problems, including pattern recognition [11], coherent structures [32–34], control theory [1,18], and model reduction for partial differential equations (PDEs). In addition, POD was incorporated in numerous numerical methods — e.g. in Galerkin method for parabolic and fluid dynamics equations [16,17], in finite differences, finite elements, finite volumes for classical parabolic problems, Navier-Stokes equations, solute transport problems [24–27], finite element format time FPDEs [22]. This approach can reduce the computational complexity. Nevertheless, to the best of author's knowledge, no work deals with the combination of POD and finite element methods for space FPDEs.

Here we develop a reduced model based on POD and a finite element method for the following problem: Find $u=u(x,y,t)$ such that

$$
\begin{aligned}
&\frac{\partial u(x,y,t)}{\partial t}-\frac{\partial^{\alpha}u(x,y,t)}{\partial|x|^{\alpha}}-\frac{\partial^{\beta}u(x,y,t)}{\partial|y|^{\beta}}=f(x,y,t), \quad (x,y,t)\in\Omega\times(0,T),\\
&u(x,y,0)=g(x,y), \quad (x,y)\in\Omega,\\
&u(x,y,t)=0, \quad (x,y)\in\mathbb{R}^2\backslash\Omega, \;\; t\in(0,T),
\end{aligned}
\tag{1.1}
$$

where $\Omega=(0,1)\times(0,1)$, $1<\alpha,\beta<2$ and $\partial^{\alpha}u/\partial|x|^{\alpha}$ denotes the Riesz fractional derivative defined by

$$
\frac{\partial^{\alpha}u}{\partial|x|^{\alpha}}:=-\frac{1}{2\cos(\alpha\pi/2)}\left({}_{-\infty}D_x^{\alpha}u+{}_xD_{+\infty}^{\alpha}u\right),
$$

and ${}_{-\infty}D_x^{\alpha}u$ and ${}_xD_{+\infty}^{\alpha}u$ are, respectively, the left- and right-sided Riemann-Liouville derivatives. Since the support of u is in Ω, it follows that

$$
{}_{-\infty}D_x^{\alpha}u(x,y,t)={}_0D_x^{\alpha}u(x,y,t), \quad {}_{-\infty}D_y^{\alpha}u(x,y,t)={}_0D_y^{\alpha}u(x,y,t), \quad (x,y)\in\Omega.
$$

A finite element method for the Eq. (1.1) is considered in Ref. [3]. We remark that the corresponding stiffness matrix is not sparse because fractional derivative is a non-local operator. Therefore, mesh refinements used to improve the accuracy, substantially increase memory requirements and computational cost.

To overcome this problem, we combine POD with a finite element method — viz. we reconstruct POD basis in the least square sense by snapshots of the solutions on uniform intervals, so that only d-element POD basis is needed to solve the Eq. (1.1). Note that d is the number of the first maximal eigenvalues of the correlation matrix G. Thus the degree of freedom is reduced and the computational time decreases substantially. This present method can be considered as an improvement of the classical finite element method.

The paper is organised as follows. In Section 2, auxiliary results are introduced. Section 3 briefly recalls classical finite element method for the Eq. (1.1). In Section 4, we choose finite element solution snapshots to construct a POD basis in the least squares optimal sense and establish a reduced finite element scheme based on POD. Section 5 deals with the stability and error estimates of the method. The efficiency of the model is demonstrated by numerical examples in Section 6.