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An Explicit Second-Order Numerical Scheme to Solve Decoupled Forward Backward Stochastic Equations

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Abstract. An explicit numerical scheme is proposed for solving decoupled forward backward stochastic differential equations (FBSDE) represented in integral equation form. A general error inequality is derived for this numerical scheme, which also implies its stability. Error estimates are given based on this inequality, showing that the explicit scheme can be second-order. Some numerical experiments are carried out to illustrate the high accuracy of the proposed scheme.

AMS subject classifications: 60H35, 65C20, 60H10

Key words: Explicit scheme, second-order, decoupled FBSDE, error estimate.

1. Introduction

We consider the numerical solution of decoupled forward backward stochastic differential equations (FBSDE) on a filtered complete probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, represented in the following equivalent integral equation form:

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s\,, \quad t \in [0, T]\,, \quad \text{(SDE)} \\ Y_t = \xi + \displaystyle\int_t^T f(s, X_s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s\,, \quad t \in [0, T]\,, \quad \text{(BSDE)} \end{cases} \tag{1.1}$$

where $\mathbb{F} = (\mathscr{F}_t)_{0 \le t \le T}$ is the natural filtration of the standard d-dimensional Brownian motion $W = (W_t)_{0 \le t \le T}$, $\mathscr{F} = \mathscr{F}_T$ with the fixed finite horizon T, $\xi \in \mathscr{F}_T$ is an L^2 integrable random variable, $b\colon \Omega \times [0, T] \times \mathbb{R}^q \to \mathbb{R}^q$, $\sigma\colon \Omega \times [0, T] \times \mathbb{R}^q \to \mathbb{R}^{q \times d}$ and $f\colon \Omega \times [0, T] \times \mathbb{R}^q \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, are all measurable functions, and $b(t, x)$, $\sigma(t, x)$ and $f(t, x, y, z)$ are $\mathscr{F}_t$-measurable for fixed $(X_t, Y_t, Z_t) = (x, y, z)$. Note that the integrals in (1.1) with respect

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to the Brownian motion W_s are of the Itô type, and the first equation arises from a standard forward stochastic differential equation (SDE), while the second arises from a backward stochastic differential equation (BSDE) since the terminal condition of $Y_T = \xi$ is given. A triple (X_t, Y_t, Z_t) is called an L^2 solution of decoupled FBSDE (1.1) if it is $\mathscr{F}_t$-adapted and square integrable.

In 1990, Pardoux and Peng proved the existence and uniqueness of the solution for nonlinear BSDE in their original work [13]. There has since been increasing attention paid to the theory of FBSDE and their application in many fields such as mathematical finance, partial differential equations (PDE), stochastic control, risk measure and game theory. However, the solution of FBSDE in closed form can seldom be found in practical problems, so numerical methods are often invoked. Under certain conditions, the relationship between the solutions of the decoupled FBSDE represented by (1.1) and parabolic PDE has led to some numerical methods to solve the FBSDE based on solving the corresponding parabolic PDE [3,7–9,11]. There are also other numerical schemes directly developed from the FBSDE [1,4,15–18].

Nevertheless, most existing highly accurate numerical methods to solve for Y_t are implicit, with a heavy computational requirement. By using the properties of the Itô integral, the trapezoidal rule and a stochastic process $\Delta\tilde{W}_{n,s}$, we propose a new explicit numerical scheme to solve the decoupled FBSDE (1.1) in the next section. Then in Section 3, after obtaining a useful inequality for the error estimate, the second-order convergence of our scheme is proved under some reasonable conditions on the coefficients of (1.1). We undertake some numerical calculations to demonstrate our theoretical results in Section 4, and give our conclusions in Section 5.

We first introduce some relevant notation as follows.

1. $|\cdot|$ denotes the standard Euclidean norm in the Euclidean space $\mathbb{R}$, $\mathbb{R}^q$ and $\mathbb{R}^{q\times d}$.
2. $L^2 = L^2_{\mathscr{F}}(0,T;\mathbb{R}^d)$ denotes the set of all $\mathscr{F}_t$-adapted and mean-square-integrable processes valued in $\mathbb{R}^d$.
3. $\mathscr{F}_s^{t,x}(t \le s \le T)$ denotes a σ-field generated by the diffusion process $\{X_r, t \le r \le s, X_t = x\}$. When $s = T$, we use $\mathscr{F}^{t,x}$ to denote $\mathscr{F}_T^{t,x}$.
4. $\mathbb{E}_s^{t,x}[\eta]$ denotes the conditional mathematical expectation of the random variable η under the σ-field $\mathscr{F}_s^{t,x}$, i.e., $\mathbb{E}_s^{t,x}[\eta] = \mathbb{E}[\eta|\mathscr{F}_s^{t,x}]$. When $s = t$, we use $\mathbb{E}_t^x[\eta]$ to denote $\mathbb{E}[\eta|\mathscr{F}_t^{t,x}]$.
5. $C_b^{l,k,k}$ denotes the set of continuously differential functions $\phi : (t,x,y) \in [0,T] \times \mathbb{R}^q \times \mathbb{R} \to \mathbb{R}$ with uniformly bounded partial derivatives $\partial_t^{l_1}\phi$ and $\partial_x^{k_1}\partial_y^{k_2}\phi$ for any positive integers $l_1 \le l$ and $k_1 + k_2 \le k$.
6. $C_b^{k_3,k_4}$ denotes the set of functions $\phi : (t,x) \in [0,T] \times \mathbb{R}^q \to \mathbb{R}$ with uniformly bounded partial derivatives up to k_3 with respect to t, and up to k_4 with respect to x.

Throughout, C stands for a generic positive constant depending only on T, the given data b, σ, f, ξ and the regularity of time partition, although its value may differ from place to place.