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Free Boundary Determination in Nonlinear Diffusion

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Abstract. Free boundary problems with nonlinear diffusion occur in various applications, such as solidification over a mould with dissimilar nonlinear thermal properties and saturated or unsaturated absorption in the soil beneath a pond. In this article, we consider a novel inverse problem where a free boundary is determined from the mass/energy specification in a well-posed one-dimensional nonlinear diffusion problem, and a stability estimate is established. The problem is recast as a nonlinear least-squares minimisation problem, which is solved numerically using the *lsqnonlin* routine from the MATLAB toolbox. Accurate and stable numerical solutions are achieved. For noisy data, instability is manifest in the derivative of the moving free surface, but not in the free surface itself nor in the concentration or temperature.

AMS subject classifications: 35R35, 35K55, 49N45, 65M06

Key words: Nonlinear diffusion, free boundary problem, finite difference method.

1. Introduction

Many industrial and scientific applications involving inverse problems have ensured that this mathematical field has undergone extensive development over several decades, including a recent emphasis on nonlinear inverse problems — e.g. the Stefan solidification problem involving nonlinear diffusion with a free boundary [1], the determination of unknown coefficients together with the temperature in a nonlinear heat conduction problem [2,3], and the procedure to find an approximate stable solution to the unknown

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coefficient from over-specified data based on the finite difference method combined with the Tikhonov regularisation approach [11].

In this article, we consider the problem of identifying the free boundary in a nonlinear diffusion problem. We formulate the inverse problem under investigation in Section 2. The respective numerical methods for solving the direct and inverse problems are described in Sections 3 and 4, and the numerical results are discussed in Section 5. Our final conclusions are presented in Section 6.

2. Mathematical Formulation

The nonlinear one-dimensional diffusion problem involves the partial differential equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(a(u)\frac{\partial u(x,t)}{\partial x}\right) + f(x,t)\,, \quad (x,t) \in \Omega \tag{2.1}$$

where the domain $\Omega = \{(x,t) : 0 < x < h(t), 0 < t < T < \infty\}$ has the unknown free smooth boundary $x = h(t) > 0$, subject to the initial condition

$$u(x,0) = \phi(x)\,, \quad 0 \leq x \leq h(0) =: h_0 \tag{2.2}$$

where $h_0 > 0$ is given and the Dirichlet boundary conditions

$$u(0,t) = \mu_1(t)\,, \quad u(h(t),t) = \mu_2(t)\,, \quad 0 \leq t \leq T\,. \tag{2.3}$$

In order to determine the unknown boundary $h(t)$ for $t \in (0,T]$, we impose the over-determination condition of integral type

$$\int_0^{h(t)} u(x,t)dx = \mu_3(t)\,, \quad 0 \leq t \leq T\,, \tag{2.4}$$

which represents the specification of mass/energy of the diffusion system [4]. The six functions $a(u) > 0$, $\phi(x)$, $\mu_i(t)$ for $i \in \{1,2,3\}$, and $f(x,t)$ are given. Physically, $u(x,t)$ represents the concentration or temperature, $a(u)$ the diffusivity, and $f(x,t)$ a source or sink. The function pair $(h(t) > 0, u(x,t)) \in C^1[0,T] \times C^{2,1}(\overline{\Omega})$ satisfying Eqs. (2.1)–(2.4) is the solution of the problem, under the following existence and uniqueness theorems [6].

Theorem 2.1. (Existence)

Assume that:

1. *$\phi(x) \in C^2[0,h_0]$, $\mu_i(t) \in C^1[0,T]$ for $i \in \{1,2,3\}$, $f(x,t) \in C^{1,0}([0,H_1]\times[0,T])$, and $a(u) \in C^1[M_0,M_1]$;*

2. *$\phi(x) > 0$ for $x \in [0,h_0]$, $\mu_i(t) > 0$ for $t \in [0,T]$, $i \in \{1,2,3\}$, $f(x,t) \geq 0$ for $(x,t) \in [0,H_1]\times[0,T]$, and $a(u) \geq a_0 > 0$ for $u \in [M_0,M_1]$ where a_0 is some given constant; and*