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Stochastic Collocation Methods via Minimisation of the Transformed L_1-Penalty

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Abstract. The sparse reconstruction of functions via a transformed ℓ_1 (TL1) minimisation is studied and theoretical results concerning recoverability and accuracy of such reconstruction from undersampled measurements are obtained. To identify the coefficients of sparse orthogonal polynomial expansions in uncertainty quantification, the method is combined with the stochastic collocation approach. The DCA-TL1 algorithm [37] is used in implementing the TL1 minimisation. Various numerical examples demonstrate the recoverability and efficiency of this method.

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Key words: Uncertainty quantification, stochastic collocation, DCA-TL1 minimisation, compressive sensing, restricted isometry property.

1. Introduction

This paper is concerned with the sparse approximation of target functions $f : \mathbb{R}^d \to \mathbb{R}, d \geq 1$, given relatively scarce training data in the frame of the generalized polynomial chaos (gPC) approach [29]. The problem occurs in uncertainty quantification with large-scaled stochastic systems, so that only limited simulations have been carried out and the number of training samples is much lower than the cardinality of the gPC basis. This leads to underdetermined systems with infinite number of solutions. Rooted in the idea of compressive sensing [4,6,8,9], stochastic collocation methods via ℓ_1-minimisation proved their efficiency in recovering sparse approximations of target functions. For more details the reader can refer to [10,14–17,19,23,30–32] and references therein.

Although ℓ_1-minimisation has attracted considerable attentions in compressive sensing, it may not perform well on some practical problems — e.g. if the measurement matrix is

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coherent, and is not able to derive the sparsest solutions. To enhance the sparsity of the solutions, various nonconvex penalties have been developed. Their metric is comparable with ℓ_1-norm but stands closer, in a sense, to ℓ_0-norm, with ℓ_p-penalties, $p \in (0,1)$ among the most popular. The recoverability of ℓ_p-minimisation with $p \in (0,1)$ is established in [27]. A stochastic collocation method via ℓ_p-minimisation to recover sparse gPC coefficients was investigated in [13]. A nonconvex ℓ_{1-2}-penalty developed in [18, 34], utilises the difference of convex algorithm (DCA) [25, 26] to obtain a sparse solution. It is worth noting that numerical experiments show the better sparsity of ℓ_{1-2} than ℓ_1. This ℓ_{1-2}-minimisation was combined with stochastic collocation for sparse gPC approximation in [33].

Here, we focus on a nonconvex transformed ℓ_1-penalty studied recently by Zhang and Xin [37]. They showed that the TL1 penalty interpolates ℓ_0 and ℓ_1 norms through a non-negative parameter $a \in (0, +\infty)$, satisfies unbiasedness, sparsity, Lipschitz continuity and developed the DCA for the TL1-minimisation problem. Moreover, they compared DCA-TL1 with with DCA for other non-convex penalties, including PiE [22], MCP [35] and SCAD [11]. It turns out that DCA-TL1 performs competitively well. Using all these results, we implement TL1 minimisation for the stochastic collocation.

The goal of this work is twofold. First, we use [3, 37] to obtain new theoretical results describing recoverability and accuracy of the TL1 minimisation reconstruction although, similarly to [33], our estimates do not demonstrate the superiority of TL1 over ℓ_1-penalty. Second, we use the stochastic collocation method via TL1 minimisation to identify the gPC expansion coefficients. In particular, we focus on Legendre polynomials in multi-dimensional case. In order to implement the constrained minimisation problem with TL1 penalty, we employ the DCA-TL1 algorithm for sparse polynomial representations introduced in [37]. In order to derive the sparsest solution in the case of non-sparse functions, we develop an adaptive approach. The algorithm, called the adaptive DCA-TL1 method, allows us to choose the parameter a in different ways. Thus, for a fixed a_i from a set a, we use the TL1 minimisation algorithm to generate the minimiser. If the minimiser, obtained by setting $a_j \in a$ and $a_j \neq a_i$ is more sparse than the previous one, we select the parameter a_j and the corresponding solution. Proceeding in this way, we will get the best parameter and the sparsest solution. Various numerical experiments demonstrate that the TL1 minimisation method is the more efficient than ℓ_1 and ℓ_{1-2} minimisations.

The rest of the paper is organised as follows. In Section 2, we introduce TL1 minimisation for stochastic collocation which is the main object of study in this paper. Section 3 is devoted to theoretical estimates in sparse and non-sparse recovery of the signals which may contain noise. The recovery via the stochastic collocation method using TL1 minimisation is discussed in Section 4. Here, we also introduce a DCA-TL1 algorithm and present its pseudo-code. The results of numerical experiments are shown in Section 5 and our concluding remarks are in Section 6.

2. gPC Expansion and Problem Setup

When a simulation model is computationally expensive to run, the approximation of the model output is often an efficient method to quantify the parametric uncertainty. The