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A Two-Grid Finite Element Method for Nonlinear Sobolev Equations

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Abstract. A two-grid based finite element method for nonlinear Sobolev equations is studied. It consists in solving small nonlinear systems related to coarse-grids, following the solution of linear systems in fine-grid spaces. The method has the same accuracy as the standard finite element method but reduces workload and saves CPU time. The H^1-error estimates show that the two-grid methods have optimal convergence if the coarse H and fine h mesh sizes satisfy the condition $h = \mathscr{O}(H^2)$. Numerical examples confirm the theoretical findings.

AMS subject classifications: 65N12, 65M60
Key words: Nonlinear Sobolev equations, two-grid finite element method, error estimate.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain with the boundary $\partial\Omega$. We consider the initial-boundary value problem

$$\begin{aligned}
&u_t - \nabla\cdot(a(u)\nabla u_t + b(u)\nabla u) = f(u), \quad (x,t) \in \Omega\times(0,T],\\
&u(x,t) = 0, \quad (x,t) \in \partial\Omega\times(0,T],\\
&u(x,0) = u_0(x), \quad x \in \Omega,
\end{aligned} \tag{1.1}$$

where $u_t := \partial u/\partial t$, $x = (x_1, x_2)$ and u_0, f are known functions and it is always assumed that $a(u)$, $b(u)$ and $f(u)$ satisfy the following conditions:

(i) The coefficients $a(u)$ and $b(u)$ are sufficiently smooth and there are constants a_i, $i = 0,1,2$ and b_i, $i = 0,1$ such that for all $u \in C(\Omega\times[0,T])$ the inequalities

$$0 < a_0 \le a(u) \le a_1, \quad |\partial a(u)/\partial t| \le a_2, \quad 0 < b_0 \le b(u) \le b_1, \tag{1.2}$$

hold.

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(ii) The functions $a(u)$, $b(u)$ and $f(u)$ satisfy the Lipschitz condition in the domain $C(\Omega \times [0,T])$ — i.e. for any $u, v \in C(\Omega \times [0,T])$ the inequalities

$$|a(u)-a(v)| \le L|u-v|, \tag{1.3}$$

$$|b(u)-b(v)| \le L|u-v|, \tag{1.4}$$

$$|f(u)-f(v)| \le L|u-v|, \tag{1.5}$$

where L is a positive constant, hold.

Nonlinear Sobolev equations arise in the flow of fluids through fissured rocks [1], thermodynamics [7], the migration of moisture in soil [24], and other applications. For the existence, uniqueness and stability of the solution, we refer the reader to Refs. [14,22]. Approximation methods for nonlinear Sobolev equations have been also considered. Thus Lin [17] studied Galerkin method for equations in two spatial dimensions, Lin and Zhang [18] established optimal L^2-error estimates for semi-discrete finite element methods, Gu [15] applied the characteristic finite element method to the semi-linear and nonlinear equation, Ohm and Lee [19] investigated fully discrete discontinuous approximations, He *et al.* [16] developed time discontinuous Galerkin space-time finite-element method, Yan *et al.* [30] considered a two-grid finite volume element method, Sun and Yang [23] a penalty discontinuous Galerkin conforming finite element method, Shi *et al.* [20, 21] a low order characteristic-nonconforming finite element method.

Two-grid method is a discretisation of nonlinear equations based on two grids of different size. The main idea is to use a coarse grid space to produce a rough approximation of the solution and then use it as the initial guess on the fine grid. This method involves a nonlinear solving on coarse grids of size H and a linear solving on fine grids of size $h < H$. The two-grid finite element method was introduced by Xu [28, 29] for solving nonsymmetric linear and nonlinear elliptic problems. Later on, two-grid methods have been investigated by many authors. Dawson and Wheeler [12, 13] applied a two-grid mixed finite element method and a two-grid finite difference method to a class of nonlinear parabolic equations. Besides, Chen *et al.* [6, 8–11] and Wu and Allen [27] used two-grid approach in expanded mixed finite element methods for semilinear and nonlinear reaction-diffusion equations, Bi and Ginting [2, 3] in the finite volume element method and the discontinuous Galerkin finite element method for the nonlinear elliptic problems, Chen *et al.* [4, 5] in finite volume element methods for semilinear parabolic equations, Weng *et al.* [25, 26] in two-level quadratic equal-order stabilised method for Stokes eigenvalue problem and in variational multiscale method with bubble stabilisation for convection diffusion equations.

Here we consider a two-grid finite element method for the nonlinear Sobolev equations. To the best of authors' knowledge, the method has not been used in the approximate solution of such equations. It is based on two conforming piecewise linear finite element spaces S_H and S_h, where S_H is the coarse grid of size H and S_h the fine grid of size h. This approach allows to split the solution of nonlinear systems for solving linear systems in the fine-grid space and a nonlinear system in a substantially smaller space. The main result of this work consists in error estimates of the method in the H^1-norm. Numerical examples confirm the theoretical estimates.