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# An Inverse Source Non-local Problem for a Mixed Type Equation with a Caputo Fractional Differential Operator

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**Abstract.** We consider the unique solvability of an inverse-source problem with integral transmitting condition for a time-fractional mixed type equation in rectangular domain where the unknown source term depends only on the space variable. The solution is based on a series expansion using a bi-orthogonal basis in space, corresponding to a non-self-adjoint boundary value problem. Under certain regularity conditions on the given data, we prove the uniqueness and existence of the solution for the given problem. The influence of the transmitting condition on the solvability of the problem is also demonstrated. Two different transmitting conditions are considered — viz. a full integral form and a special case. In order to simplify the bulky expressions appearing in the proof of our main result, we establish a new property of the recently introduced Mittag-Leffler type function in two variables.

**AMS subject classifications**: 35M10, 35R30

**Key words**: Inverse-source problem, mixed type equation, Caputo fractional operator.

## 1. Introduction

Fractional differential equations have become important in models of many real life processes arising in different fields such as water movement in soil [1], nanotechnology [2] and viscoelasticity [3]. In particular, inverse problems include identifying the coefficients or order in fractional differential equations, or associated boundary conditions or source functions as in this article. Li *et al.* [4] dealt with an inverse problem where they simultaneously identified the space-dependent diffusion coefficient and the fractional order in the 1D time-fractional diffusion equation. An inverse-source problem may involve looking for the source

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term(s) in a differential equation (or a system of equations) using extra boundary data. Kirane *et al.* [5] studied conditional well-posedness in determining a space-dependent source in the 2D time-fractional diffusion equation, and Aleroev *et al.* [6] obtained the source term for a time fractional diffusion equation with an integral type over-determining condition.

Inverse problems for various mixed type equations of integer order were studied by Sabitov and his students in several publications. For instance, the Lavrent'ev-Bitsadze equation

$$u_{xx} + \operatorname{sgn}(y) u_{yy} = f(x)$$

was considered in a rectangular domain, when the unique solvability of the inverse problem with non-local conditions with respect to space variables was proven using the method of spectral expansions [7]. Other authors considered well-posedness and stability inequalities for the solution of three source identification problems for hyperbolic-parabolic equations [8]. In Ref. [9], inverse problems for time-fractional mixed type equations with uniformly elliptic operator with respect to space variable were studied in a rectangular domain, where a unique weak solvability was proven under certain regularity conditions on the given functions and geometric restrictions on the domain. The influence of transmitting conditions in mixed type equations is interesting, since restrictions on the given data could be reduced depending on the form of the transmitting conditions — cf. Ref. [10] for the fractional case and Ref. [11] for the integer case of mixed type equations.

An inverse source problem with non-local conditions for a time-fractional mixed type equation with Caputo derivative is considered in this article. Given a non-local condition, we use a series expansion on a bi-orthogonal basis corresponding to a non-self-adjoint boundary value problem. Such a bi-orthogonal basis was used in Ref. [12], where the equation considered contains a generalised fractional derivative. Section 2 summarises the Gamma and Beta functions, Mittag-Leffler type functions, and the Riemann-Liouville and Caputo fractional differential operators, and includes a short commentary on the bi-orthogonal system. Section 3 is devoted to the problem formulation and construction of our formal solution. We prove the existence and uniqueness of the solution for the case $0 < \gamma < 1$ in Section 4, and consider the special case of the transmitting condition $\gamma = 1$ in Section 5, where $\gamma$ denotes the exponent of the fractional derivative of interest. Our conclusions are in Section 6, and some supporting detailed calculations are presented in the Appendix.

## 2. Preliminaries

### 2.1. Euler integrals

The Gamma function is the Euler integral [13, p.24]:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

For $z \in \mathbb{R}^+$, this integral converges and we have the familiar recurrence relation

$$\Gamma(z+1) = z\Gamma(z) , \tag{2.1}$$