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Sparse Grid Collocation Method for an Optimal Control Problem Involving a Stochastic Partial Differential Equation with Random Inputs

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Abstract. In this article, we propose and analyse a sparse grid collocation method to solve an optimal control problem involving an elliptic partial differential equation with random coefficients and forcing terms. The input data are assumed to be dependent on a finite number of random variables. We prove that an optimal solution exists, and derive an optimality system. A Galerkin approximation in physical space and a sparse grid collocation in the probability space is used. Error estimates for a fully discrete solution using an appropriate norm are provided, and we analyse the computational efficiency. Computational evidence complements the present theory, to show the effectiveness of our stochastic collocation method.

AMS subject classifications: 49A22, 49B22, 65M55, 65N30

Key words: Sparse grid collocation, stochastic partial differential equation, distributed control, finite element methods.

1. Introduction

We seek a solution (u,f) that minimises the cost functional

$$\mathscr{J}(u,f)=\mathbb{E}\left(\frac{1}{2}\int_D |u-U|^2\,dx+\frac{\beta}{2}\int_D |f|^2\,dx\right) \tag{1.1}$$

and satisfies the stochastic elliptic problem involving a Dirichlet boundary condition:

$$\begin{aligned} &-\nabla\cdot[a(x,\omega)\nabla u(x,\omega)]=f(x,\omega) \quad \text{in } D\times\Omega,\\ &u(x,\omega)=0 \quad \text{on } \partial D\times\Omega, \end{aligned} \tag{1.2}$$

where $\mathbb{E}$ denotes expected value, D the spatial domain and ∂D its boundary, U a target solution to the constraint, β a positive constant influencing the relative importance of the two terms in Eq. (1.1), and f a stochastic control acting in the domain and depending on $a(x,\omega)$. The stochastic elliptic PDE generally models fluid flow in porous media; and under the homogeneous Dirichlet boundary condition, for almost every $\omega\in\Omega$ we look for a

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solution u that is a stochastic function from $\overline{D} \times \Omega$ to $\mathbb{R}$, where $D \subset \mathbb{R}^d$ is a convex bounded polygonal domain, and $a : D \times \Omega \to \mathbb{R}$ is a stochastic function. Here a has a bounded continuous covariance function in the context of a Karhunen-Loève (KL) expansion, with a uniformly bounded continuous first derivative, and ∇ means differentiation with respect to $x \in D$ only. This type of stochastic elliptic problem was previously investigated in the articles [2, 3, 12, 13, 22] and references therein.

To analyse this stochastic optimal control problem, we first estimate the error of the solution to the stochastic partial differential equation (SPDE), and then use the Brezzi-Rappaz-Raviart (BRR) theory to produce an error estimate of the solution to the stochastic optimal control problem. We then construct a computational algorithm for our stochastic control problem and present some numerical examples with a given target solution to the stochastic optimal control problem with a distributed control in the domain. In order to solve the stochastic optimal control problem numerically, we adopt a stochastic collocation method that has gained much attention recently in the computational community [4, 5]. Stochastic collocation can be based on either full or sparse tensor product approximation spaces, and seems to be ideal for computing statistics from solutions of PDE with random input data, since it essentially preserves the fast convergence of the spectral Galerkin method in maintaining an ensemble based approach (just as Monte Carlo). On the other hand, approximations based on tensor product grids suffer from the curse of dimensionality, since the number of collocations in a tensor grid grows exponentially fast in the number of input random variables. Thus even if the number of random variables is only moderately large, one should consider sparse tensor product spaces as first proposed by Smolyak [23]. Recently, total degree polynomial spaces and sparse tensor product spaces were investigated [8, 14, 24, 25]; and there have been substantial developments in stochastic collocation methods since [4, 5, 20, 21], where effective collocation strategies for problems involving a moderately large number of random variables have been devised.

The plan of this article is as follows. We represent a random field in Section 2, introducing the KL expansion and its truncated expansion. We also analyse our constraint equation and stochastic elliptic PDE, transforming a stochastic problem to a high-dimensional deterministic problem and presenting *a priori* error estimates. In Section 3, we introduce the discretisation method for probability space — viz. a sparse grid collocation method. In Section 4, the optimality system of equations is derived, showing the existence of a unique minimiser. We then establish our error estimate for the discrete approximate solutions to the optimality system, and in Section 5 give two numerical examples of stochastic optimal control problems constrained by the stochastic elliptic PDE under the Dirichlet boundary condition. Our brief concluding remarks are made in the final Section 6.

2. Preliminaries

2.1. Function spaces and problem setting

For our stochastic elliptic problem, we use a complete probability space $(\Omega, \mathscr{F}, P)$ where Ω is a set of outcomes, $\mathscr{F}$ is a σ-algebra of events, and $P : \mathscr{F} \to [0, 1]$ is a probability