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# A Subspace-Projected Approximate Matrix Method for Systems of Linear Equations

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**Abstract.** Given two $n\times n$ matrices $A$ and $A_0$ and a sequence of subspaces $\{0\}=\mathcal{V}_0\subset\cdots\subset\mathcal{V}_n=\mathbb{R}^n$ with $\dim(\mathcal{V}_k)=k$, the $k$-th *subspace-projected approximated matrix* $A_k$ is defined as $A_k=A+\Pi_k(A_0-A)\Pi_k$, where $\Pi_k$ is the orthogonal projection on $\mathcal{V}_k^\perp$. Consequently, $A_kv=Av$ and $v^*A_k=v^*A$ for all $v\in\mathcal{V}_k$. Thus $(A_k)_{k\geq 0}^n$ is a sequence of matrices that gradually changes from $A_0$ into $A_n=A$. In principle, the definition of $\mathcal{V}_{k+1}$ may depend on properties of $A_k$, which can be exploited to try to force $A_{k+1}$ to be closer to $A$ in some specific sense. By choosing $A_0$ as a simple approximation of $A$, this turns the subspace-approximated matrices into interesting preconditioners for linear algebra problems involving $A$. In the context of eigenvalue problems, they appeared in this role in Shepard *et al.* (2001), resulting in their *Subspace Projected Approximate Matrix method*. In this article, we investigate their use in solving linear systems of equations $Ax=b$. In particular, we seek conditions under which the solutions $x_k$ of the approximate systems $A_kx_k=b$ are computable at low computational cost, so the efficiency of the corresponding method is competitive with existing methods such as the Conjugate Gradient and the Minimal Residual methods. We also consider how well the sequence $(x_k)_{k\geq 0}$ approximates $x$, by performing some illustrative numerical tests.

**AMS subject classifications**: 65F10, 65F08

**Key words**: Linear system, Galerkin method, subspace projected approximate matrix, minimal residual method.

## 1. Introduction

Subspace iterative methods for linear algebra problems are based on the repeated application of two consecutive ideologically separate steps — viz. the generic *selection* of a suitable approximation from the space, followed by increasing its dimension in an *expansion* of the subspace. In this section, we recall two well-known selection mechanisms in the context of linear systems — viz. the *Ritz-Galerkin* and *Minimal Residual* procedures. We then suggest an alternative.

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## 1.1. Selecting approximations from a subspace $\mathcal{V}$

In the context of solving a linear system $Ax = b$ and a given a subspace $\mathcal{V} \subset \mathbb{C}^n$, two well known and successful selection mechanisms are the *Ritz-Galerkin* approximation

$$\textit{Ritz-Galerkin:}\ \text{Find } \hat{x} \in \mathcal{V} \text{ such that } b - A\hat{x} \perp \mathcal{V}, \tag{1.1}$$

and the *Minimal Residual* approximation

$$\textit{Minimal Residual:}\ \text{Find } \hat{x} \in \mathcal{V} \text{ for which } \|b - A\hat{x}\| \text{ is minimal.} \tag{1.2}$$

On choosing a matrix $V$ with columns spanning $\mathcal{V}$ and denoting the transpose by the superscript $t$, the Ritz-Galerkin approximation can be computed by solving

$$\hat{x} = Vy \quad \text{where} \quad V^t(b - AVy) = 0\,, \tag{1.3}$$

whereas the Minimal Residual approximation can be computed by solving

$$\hat{x} = Vy \quad \text{where} \quad (AV)^t(b - AVy) = 0\,, \tag{1.4}$$

because the minimum in (1.2) is realised by the $\hat{x} \in \mathcal{V}$ for which $A\hat{x}$ equals the orthogonal projection of $b$ on $A\mathcal{V}$. Although neither approximation $\hat{x}$ depends on the actual choice of the basis $V$ for $\mathcal{V}$, the basis is of interest for efficient implementation of the method. For example, if the matrix $A$ is symmetric and positive definite, in the Ritz-Galerkin approach the basis can be chosen to be $A$-orthonormal — i.e. such that $V^tAV = I$ and consequently $\hat{x} = Vy = VV^tb$. If the space $\mathcal{V}$ is then *expanded* by appending another $A$-orthonormal basis vector $v$ to $V$, the new approximation differs only by a simple update $vv^tb$ from the previous one. However, this elegant outcome should not deter us from considering other options for the basis for $\mathcal{V}$.

**Remark 1.1.** If the spaces $(\mathcal{V}_k)_{k=0}^n$ form a sequence of Krylov subspaces, the Ritz-Galerkin approach leads to the *Conjugate Gradient method* [9] when $A$ is symmetric and positive definite, and the *Full Orthogonalization method* for general non-symmetric $A$. The Minimal Residual approach leads to the *MinRES* [12] method if $A$ is symmetric, and to the *GMRES* [16] method for non-symmetric $A$. The Ritz-Galerkin approach is also used in *coarse grid corrections* within the *MultiGrid* method. Both the Ritz-Galerkin and the Minimal Residual approach are also used in finite element methods to approximately solve partial differential equations.

## 1.2. Selecting an approximation associated with a subspace $\mathcal{V}$

The Ritz-Galerkin and the Minimal Residual approach define aproximations *v from* the space $\mathcal{V}$, in the sense that $\hat{x} \in \mathcal{V}$. If the space has dimension $k << n$, not more than $k$ matrix-vector products (MVPs) with $A$ are needed to compute these approximations. Since the number of MVPs with $A$ is often a good indication of the cost of a subspace iterative