East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.110312.240412a

Vol. **2**, No. 2, pp. 170-184
May 2012

A High-Order Difference Scheme for the Generalized Cattaneo Equation

Seak-Weng Vong[1,*], Hong-Kui Pang[2] and Xiao-Qing Jin[1]

[1] *Department of Mathematics, University of Macau, Macao, P.R. China.*
[2] *School of Mathematical Sciences, Jiangsu Normal University, Xuzhou, P.R. China.*

Received 11 March 2012; Accepted (in revised version) 24 April 2012

Available online 27 April 2012

Abstract. A high-order finite difference scheme for the fractional Cattaneo equation is investigated. The L_1 approximation is invoked for the time fractional part, and a compact difference scheme is applied to approximate the second-order space derivative. The stability and convergence rate are discussed in the maximum norm by the energy method. Numerical examples are provided to verify the effectiveness and accuracy of the proposed difference scheme.

AMS subject classifications: 65M06, 65M12, 65M15, 35Q51

Key words: Fractional Cattaneo equation, L_1 approximation, compact finite difference, stability, convergence.

1. Introduction

In this paper, we consider the numerical solution of a generalized Cattaneo equation [5,17] with a non-homogeneous term $f(x,t)$:

$$\frac{\partial u(x,t)}{\partial t}+\gamma\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=D\frac{\partial^2 u(x,t)}{\partial x^2}+f(x,t)\,, \tag{1.1}$$

where γ is a nonnegative constant related to the relaxation time, D is the diffusion constant, and $f(x,t)$ is a known function. The notation $\partial^\alpha/\partial t^\alpha$ in (1.1) denotes the time fractional derivative operator based on Caputo's definition [9,16], given by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\equiv\frac{1}{\Gamma(2-\alpha)}\int_0^t\frac{\partial^2 u(x,s)}{\partial s^2}\frac{ds}{(t-s)^{\alpha-1}}\,,\qquad \alpha\in(1,2)\,, \tag{1.2}$$

where $\Gamma(\cdot)$ is the gamma function. The standard Cattaneo equation is normally obtained by using a generalized form of Fick's law [8]. The equation describes a diffusion process

*Corresponding author. *Email addresses:* `swvong@umac.mo` (S.-W. Vong), `panghongkui@163.com` (H.-K. Pang), `xqjin@umac.mo` (X.-Q. Jin)

with a finite velocity of propagation and has a variety of applications — e.g. extended irreversible thermodynamics [8], modelling both heat and mass transfer [7], (inflationary) cosmological models [27], and the diffusion theory in crystalline solids [6]. However, the classical Cattaneo equation cannot describe the anomalous diffusion behavior observed in many natural systems. To address this issue, Compte and Metzler [1] generalized the classical Cattaneo model to the time fractional Cattaneo models and studied the properties of the corresponding fractional Cattaneo equations in both the long-time and short-time limits. Following Compte and Metzler, Kosztołowicz and Lewandowska [10] presented a theoretical foundation for studies of the subdiffusion impedance using a hyperbolic equation. Povstenko [17] considered the generalized Cattaneo-type equations with time fractional derivatives and formulated the corresponding theory of thermal stresses. Qi and Jiang [18] extended the classical Cattaneo equation to the space-time fractional Cattaneo equation and derived the exact solution by joint Laplace and Fourier transforms.

Although some theoretical analysis has been presented for the generalized Cattaneo equations [1, 17, 18], little work has been done on numerical methods. Currently, Ghazizadeh *et al.* [5] derive the generalized Cattaneo equation using a concept of single-phase lag equation [25] and a recently introduced fractional Taylor series expansion formula [15]. Two finite difference schemes, namely an explicit predictor-corrector scheme and a totally implicit scheme, have been developed [5]. In recent years, some numerical methods have been proposed for solving other types of fractional differential equations. Meerschaert and Tadjeran [13,14] investigated space-fractional differential equations, and proposed an implicit Euler method based on a shifted Grünwald formula to approximate fractional derivatives of order $1 < \alpha < 2$. Yuste and Acedo [26] proposed an explicit finite difference method and analyzed the stability condition for the fractional subdiffusion equation. Langlands and Henry [11] also considered this type of equation, and constructed an implicit finite difference by using the L_1 scheme to approximate the fractional derivative. The accuracy and stability were discussed by the Fourier method. Zhuang *et al.* [28] studied the stability and convergence of an implicit numerical method by the energy method. Cui [2] used a fourth-order compact difference scheme to increase the spatial accuracy for solving the fractional anomalous subdiffusion equation with a nonhomogeneous term. Du *et al.* [3] derived a compact difference scheme for the fractional diffusion-wave equation based on the L_1 approximation. Gao and Sun [4] first transformed the original fractional subdiffusion problem to an equivalent form and then applied the compact difference scheme with the L_1 approximation to discretize the resulting equation. By introducing a new inner product, they analyzed the stability and convergence of the proposed scheme by the energy method. For relevant main elements and ideas, reference can be made to the original papers in Refs. [12, 23].

We consider the numerical solution of the generalized Cattaneo equation (1.1). In Section 2, two new variables are introduced to transform the original equation (1.1) into a low order system of equations (cf. [21]), and the numerical solution of the low order equation is then investigated by applying the L_1 approximation to the time fractional derivative and the compact difference scheme to the second-order space derivative (cf. [3,4]). Theoretical analysis in Section 3 shows that the resulting difference scheme is unconditionally stable