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A Posteriori Error Estimates of Lowest Order Raviart-Thomas Mixed Finite Element Methods for Bilinear Optimal Control Problems

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Abstract. A Raviart-Thomas mixed finite element discretization for general bilinear optimal control problems is discussed. The state and co-state are approximated by lowest order Raviart-Thomas mixed finite element spaces, and the control is discretized by piecewise constant functions. *A posteriori* error estimates are derived for both the coupled state and the control solutions, and the error estimators can be used to construct more efficient adaptive finite element approximations for bilinear optimal control problems. An adaptive algorithm to guide the mesh refinement is also provided. Finally, we present a numerical example to demonstrate our theoretical results.

AMS subject classifications: 49J20, 65N30

Key words: Bilinear optimal control problems, lowest order Raviart-Thomas mixed finite element methods, a posteriori error estimates, adaptive algorithm.

1. Introduction

Optimal control problems with various physical backgrounds arise in many practical scientific areas, and efficient numerical methods can substantially reduce associated computational work. Two early papers devoted to finite element methods for linear elliptic optimal control problems studied error estimates for the numerical discretization [14, 15], the finite element approach for a class of constrained nonlinear optimal control problems has been considered [16], and *a posteriori* error estimates for the finite element approximation of nonlinear elliptic optimal control problems have also been derived [21]. Adaptive

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finite element methods are now widely used in engineering simulations. Efficient adaptive finite element methods can greatly reduce discretization errors, and there have been recent developments for optimal control problems. Some basic concepts have been introduced for adaptive finite element discretization in optimal control problems involving partial differential equations [5], and *a posteriori* error estimates for distributed elliptic optimal control problems have been obtained [19]. Recently, Feng has discussed an adaptive finite element method for the estimation of distributed parameters in elliptic equations [13]. All of this work addresses the standard finite element method.

In many control problems, the objective functional contains the gradient of the state variables, which should be rendered accurately in the numerical discretization of the coupled state equations. Mixed finite element methods are appropriate for the state equations in such cases, since both the scalar variable and its flux variable can then be approximated to the same accuracy. Many important contributions have been made to aspects of the mixed finite element method for linear optimal control problems, but there is not much theoretical analysis of mixed finite element approximations for bilinear or strong nonlinear optimal control problems. However, *a priori* error estimates and superconvergence for linear optimal control problems using mixed finite element methods have been obtained [9,11,12,23], and also *a posteriori* error estimates of mixed finite element methods for quadratic elliptic optimal control problems [10].

The mixed finite element approximation for quadratic optimal control problems governed by semilinear elliptic equation has previously been discussed, including *a posteriori* error estimates for the mixed finite element solution [22]. In this paper, we consider adaptive mixed finite element methods for bilinear optimal control problems. We adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω, with a norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p=\sum_{|\alpha|\le m}\|D^\alpha v\|_{L^p(\Omega)}^p$ and a semi-norm $|\cdot|_{m,p}$ given by $|v|_{m,p}^p=\sum_{|\alpha|=m}\|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega)=\{v\in W^{m,p}(\Omega):v|_{\partial\Omega}=0\}$. For p=2, we denote $H^m(\Omega)=W^{m,2}(\Omega)$, $H_0^m(\Omega)=W_0^{m,2}(\Omega)$, and $\|\cdot\|_m=\|\cdot\|_{m,2}$, $\|\cdot\|=\|\cdot\|_{0,2}$.

The general form of the bilinear optimal control problems of interest is

$$\min_{u\in K\subset U}\{g_1(\boldsymbol{p})+g_2(y)+j(u)\} \tag{1.1}$$

subject to the state equation

$$\begin{aligned} &\operatorname{div}\boldsymbol{p}+yu=f, && x\in\Omega, && (1.2)\\ &\boldsymbol{p}=-A\nabla y, && x\in\Omega, && (1.3)\\ &y=0, && x\in\partial\Omega\,, && (1.4)\end{aligned}$$

where $\Omega\subset\mathbf{R}^2$ is a bounded open set with boundary $\partial\Omega$ and $f\in L^2(\Omega)$. The 2×2 coefficient matrix $A(x)=(a_{i,j}(x))_{2\times2}\in L^\infty(\Omega;\mathbf{R}^{2\times2})$ is symmetric, and there is a constant $c>0$ such that $\mathbf{X}'A\mathbf{X}\ge c\|\mathbf{X}\|_{\mathbf{R}^2}^2$ for any vector $\mathbf{X}\in\mathbf{R}^2$. Furthermore, we assume that K is a closed