An Accelerated SOR-Like Method for Generalised Saddle Point Problems

Ting-Ting Feng[1], Guo-Liang Chen[1] and Xue-Ping Guo[1,*]

[1] *Department of Mathematics, Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice, East China Normal University, Shanghai, China.*

Abstract. A recent accelerated SOR-like method for generalised saddle point problems is discussed. Sufficient conditions for convergence are derived, and some numerical experiments illustrate its effectiveness.

1. Introduction

Generalised saddle point problems arise in constrained quadratic programming, constrained least squares problems, mixed finite-element approximations of elliptic PDEs, computational fluid dynamics, and Stokes problems [1, 6, 9, 10, 22]. Let m, n be integers such that $m \geq n > 0$. We consider the generalised saddle point problem

$$\begin{pmatrix} A & B \\ B^\mathsf{T} & -C \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{q} \end{pmatrix}, \tag{1.1}$$

where $A \in \mathbb{R}^{m \times m}$ and $C \in \mathbb{R}^{n \times n}$ are respectively symmetric positive definite and symmetric positive semi-definite matrices, B^T is the transpose of a full column rank matrix $B \in \mathbb{R}^{m \times n}$, and $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{q} \in \mathbb{R}^n$ are given vectors. In the special case $C = 0$, the problem (1.1) obviously is reduced to the augmented system of linear equations

$$\begin{pmatrix} A & B \\ B^\mathsf{T} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{q} \end{pmatrix}. \tag{1.2}$$

Various iteration methods have been used to solve such problems — e.g. Uzawa-type methods [8, 11, 12, 19, 25], Hermitian and skew-Hermitian splitting (HSS) iteration [2–5], preconditioned Krylov subspace methods [1, 21], restrictively preconditioned conjugate

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*Corresponding author. *Email addresses:* `tofengtingting@163.com` (T.-T. Feng), `glchen@math.ecnu.edu.cn` (G.-L. Chen), `xpguo@math.ecnu.edu.cn` (X.-P. Guo)

gradient methods [6,23] and the successive overrelaxation (SOR) method [24], while Benzi *et al.* [10] reviewed existing approaches. For the augmented linear system (1.2), Golub *et al.* [16] proposed an SOR-like method, which was then further developed [7,13,14]. For the generalised saddle point problem (1.1), Cao [12] discussed the convergence of a nonlinear Uzawa algorithm; Bai & Wang [8] studied parameterised inexact Uzawa methods (PIU), Zhou & Zhang [25] proposed a generalisation of the parameterised inexact Uzawa methods (GPIU), and Huang & Ma [18] developed a new GSOR method. Refs. [8,18,25] deal with symmetric positive definite matrices C, Miao & Cao [19] discussed the GPIU method [25] for symmetric positive semidefinite matrices C under the conditions $\ker(C) \cap \ker(B^\mathsf{T}) = 0$ with the rank p of C such that $0 < p < n$.

Recently, Njeru & Guo [20] have considered an accelerated SOR-like method (ASOR) for the augmented linear system (1.2). Here we apply the ASOR method to the generalised saddle point problem (1.1) when C is a symmetric positive semidefinite matrix, consider properties of the eigenpairs of the iteration matrix, and establish sufficient convergence conditions for this method. In Section 2, we introduce the ASOR method for the problem (1.2), and discuss convergence for the generalised saddle point problem (1.1) in Section 3. Numerical experiments presented in Section 4 illustrate the efficiency of the method.

2. The ASOR Method

Let us rewrite the augmented linear system (1.2) as

$$\begin{pmatrix} A & B \\ -B^\mathsf{T} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ -\mathbf{q} \end{pmatrix}$$

with the coefficient matrix

$$\mathscr{A} = \begin{pmatrix} A & B \\ -B^\mathsf{T} & 0 \end{pmatrix} = D - L - U,$$

where

$$D = \begin{pmatrix} \alpha A & 0 \\ 0 & Q \end{pmatrix}, \quad L = \begin{pmatrix} -A & 0 \\ B^\mathsf{T} & \frac{1}{2}Q \end{pmatrix}, \quad U = \begin{pmatrix} \alpha A & -B \\ 0 & \frac{1}{2}Q \end{pmatrix},$$

involve a positive number α and a symmetric positive definite matrix $Q \in \mathbb{R}^{n \times n}$.

Let ω be a positive number. In the ASOR method we seek the solution of the problem (1.2) by the iteration scheme

$$\mathbf{x}^{(k+1)} = \frac{\alpha}{\alpha + \omega} \mathbf{x}^{(k)} - \frac{\omega}{\alpha + \omega} A^{-1}(B\mathbf{y}^{(k)} - \mathbf{b}),$$

$$\mathbf{y}^{(k+1)} = \mathbf{y}^{(k)} + \frac{2\omega}{2 - \omega} Q^{-1}(B^\mathsf{T}\mathbf{x}^{(k+1)} - \mathbf{q}),$$

with the iteration matrix

$$M_{\alpha,\omega} = \begin{pmatrix} \dfrac{\alpha}{\alpha+\omega}I_m & \dfrac{-\omega}{\alpha+\omega}A^{-1}B \\ \dfrac{2\alpha\omega}{(\alpha+\omega)(2-\omega)}Q^{-1}B^{\intercal} & I_n - \dfrac{2\omega^2}{(\alpha+\omega)(2-\omega)}Q^{-1}B^{\intercal}A^{-1}B \end{pmatrix}.$$

As mentioned, this method was proposed by Njeru & Guo [20], and was compared in the numerical experiments with SOR-like [16], GSOR [7] and GSSOR [13] methods. It was shown that the method converges if and only if $0 < \omega < 2$ and $\omega^2 \mu_{max}/(2-\omega) < 2\alpha + \omega$, where μ_{max} is the largest eigenvalue of the matrix $Q^{-1}B^{\intercal}A^{-1}B$.

3. The ASOR Method for the Generalised Saddle Point Problem and Its Convergence

In this section, we study the ASOR method for the generalised saddle problem (1.1) and provide sufficient conditions for its convergence.

3.1. The ASOR method for the problem (1.1)

Analogously to Section 2, we rewrite the problem (1.2) as

$$\mathscr{A}\mathbf{z} := \begin{pmatrix} A & B \\ -B^{\intercal} & C \end{pmatrix}\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ -\mathbf{q} \end{pmatrix}, \tag{3.1}$$

and represent the matrix $\mathscr{A}$ in the form $\mathscr{A} = D - L - U$, where

$$D = \begin{pmatrix} \alpha A & 0 \\ 0 & Q \end{pmatrix}, \quad L = \begin{pmatrix} -A & 0 \\ B^{\intercal} & \frac{1}{2}Q \end{pmatrix}, \quad U = \begin{pmatrix} \alpha A & -B \\ 0 & \frac{1}{2}Q - C \end{pmatrix}, \tag{3.2}$$

with a positive number α and a symmetric positive definite matrix $Q \in \mathbb{R}^{n \times n}$.

Let ω be a positive number. The ASOR scheme for the generalised saddle problem (3.1) is

$$\begin{pmatrix} \mathbf{x}^{(k+1)} \\ \mathbf{y}^{(k+1)} \end{pmatrix} = \mathscr{M}_{\alpha,\omega}\begin{pmatrix} \mathbf{x}^{(k)} \\ \mathbf{y}^{(k)} \end{pmatrix} + \omega(D - \omega L)^{-1}\begin{pmatrix} \mathbf{b} \\ -\mathbf{q} \end{pmatrix},$$

with the iteration matrix

$$\mathscr{M}_{\alpha,\omega} = (D - \omega L)^{-1}((1-\omega)D + \omega U)$$

$$= \begin{pmatrix} (\alpha+\omega)A & 0 \\ -\omega B^{\intercal} & \dfrac{2-\omega}{2}Q \end{pmatrix}^{-1}\begin{pmatrix} \alpha A & -\omega B \\ 0 & \dfrac{2-\omega}{2}Q - \omega C \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1}{\alpha+\omega}A^{-1} & 0 \\ \dfrac{2\omega}{(\alpha+\omega)(2-\omega)}Q^{-1}B^{\intercal}A^{-1} & \dfrac{2}{2-\omega}Q^{-1} \end{pmatrix}\begin{pmatrix} \alpha A & -\omega B \\ 0 & \dfrac{2-\omega}{2}Q - \omega C \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\alpha}{\alpha+\omega}I_m & \dfrac{-\omega}{\alpha+\omega}A^{-1}B \\ \dfrac{2\alpha\omega}{(\alpha+\omega)(2-\omega)}Q^{-1}B^{\intercal} & I_n - \dfrac{2\omega^2}{(\alpha+\omega)(2-\omega)}Q^{-1}B^{\intercal}A^{-1}B - \dfrac{2\omega}{2-\omega}Q^{-1}C \end{pmatrix}.$$

Consequently, the ASOR iteration scheme for the problem (1.1) is

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \frac{\omega}{\alpha + \omega} A^{-1}(\mathbf{b} - A\mathbf{x}^{(k)} - B\mathbf{y}^{(k)}),$$

$$\mathbf{y}^{(k+1)} = \mathbf{y}^{(k)} + \frac{2\omega}{2 - \omega} Q^{-1}(B^{\mathsf{T}}\mathbf{x}^{(k+1)} - C\mathbf{y}^{(k)} - \mathbf{q}). \tag{3.3}$$

Let us consider the matrix

$$\mathcal{H}_{\alpha,\omega} = \frac{1}{\omega}(D - \omega L) = \begin{pmatrix} \dfrac{\alpha + \omega}{\omega} A & 0 \\ -B^{\mathsf{T}} & \dfrac{2 - \omega}{2\omega} Q \end{pmatrix}.$$

Recalling symmetry and positive definiteness of matrices A and Q, we note that $D - \omega L$ is invertible if and only if $\alpha + \omega \neq 0$ and $\omega \neq 2$. Now let us write

$$\mathcal{A} = \mathcal{H}_{\alpha,\omega} - \mathcal{N}_{\alpha,\omega},$$

where

$$\mathcal{N}_{\alpha,\omega} = \begin{pmatrix} \dfrac{\alpha}{\omega} A & -B \\ 0 & \dfrac{2 - \omega}{2\omega} Q - C \end{pmatrix}.$$

It is readily seen that

$$\mathcal{M}_{\alpha,\omega} = \mathcal{H}_{\alpha,\omega}^{-1} \mathcal{N}_{\alpha,\omega}, \tag{3.4}$$

so the ASOR method converges if and only if the spectral radius $\rho(\mathcal{M}_{\alpha,\omega})$ of the matrix $\mathcal{M}_{\alpha,\omega}$ satisfies the inequality

$$\rho(\mathcal{M}_{\alpha,\omega}) < 1.$$

3.2. Convergence of the ASOR method for the problem (1.1).

Here we present sufficient conditions for the convergence of the ASOR method applied to the generalised saddle point problem (1.1). To do so, we consider the eigenpairs of the iteration matrix $\mathcal{M}_{\alpha,\omega}$.

Theorem 3.1. *Assume that $A \in \mathbb{R}^{m \times m}$, $Q \in \mathbb{R}^{n \times n}$ are symmetric positive definite matrices, $B \in \mathbb{R}^{m \times n}$ is a full column rank matrix, and $\mathbf{w} := (\mathbf{u}^*, \mathbf{v}^*)^* \in \mathbb{C}^{m+n}$ is an eigenvector corresponding to the eigenvalue λ of the iteration matrix $\mathcal{M}_{\alpha,\omega}$. Then $\lambda(\alpha + \omega) = \alpha$ or*

$$\lambda^2 + \left[\frac{2\omega^2 \gamma}{(2-\omega)(\alpha+\omega)} - \frac{2\alpha+\omega}{\alpha+\omega} + \frac{2\omega}{2-\omega}\eta \right]\lambda + \left(1 - \frac{2\omega}{2-\omega}\eta\right)\left(\frac{\alpha}{\alpha+\omega}\right) = 0, \tag{3.5}$$

where

$$\eta := \eta(\mathbf{v}) = \frac{\mathbf{v}^* C \mathbf{v}}{\mathbf{v}^* Q \mathbf{v}}, \qquad \gamma := \gamma(\mathbf{v}) = \frac{\mathbf{v}^* B^{\mathsf{T}} A^{-1} B \mathbf{v}}{\mathbf{v}^* Q \mathbf{v}}. \tag{3.6}$$

Proof. The relation (3.4) implies

$$\mathcal{N}_{\alpha,\omega}\mathbf{w} = \lambda\mathcal{H}_{\alpha,\omega}\mathbf{w}\,,$$

or

$$\begin{pmatrix} \dfrac{\alpha}{\omega}A & -B \\ 0 & \dfrac{2-\omega}{2\omega}Q-C \end{pmatrix}\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \lambda\begin{pmatrix} \dfrac{\alpha+\omega}{\omega}A & 0 \\ -B^{\mathsf{T}} & \dfrac{2-\omega}{2\omega}Q \end{pmatrix}\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}\,,$$

which leads to the system of equations

$$\begin{aligned} (\alpha-\lambda\alpha-\lambda\omega)A\mathbf{u} &= \omega B\mathbf{v}\,, \\ \big((1-\lambda)(2-\omega)Q-2\omega C\big)\mathbf{v} &= -2\omega\lambda B^{\mathsf{T}}\mathbf{u}\,. \end{aligned} \qquad (3.7)$$

We note that $\lambda \neq 1$ and $\mathbf{u} \neq 0$. Indeed, assuming $\lambda = 1$ we obtain

$$-A\mathbf{u} = B\mathbf{v}\,, \quad C\mathbf{v} = B^{\mathsf{T}}\mathbf{u}\,,$$

and simple transformations show that

$$(C+B^{\mathsf{T}}A^{-1}B)\mathbf{v} = 0\,.$$

Now $C+B^{\mathsf{T}}A^{-1}B$ is a symmetric positive definite matrix, therefore the solution $\mathbf{v}$ of this homogeneous equation is trivial, and consequently the vector $\mathbf{u} = -A^{-1}B\mathbf{v} = 0$ also. This contradicts the assumption that $\mathbf{w}$ is an eigenvector of $\mathcal{M}_{\alpha,\omega}$, hence $\lambda \neq 1$. The result $\mathbf{u} \neq 0$ can be proven similarly.

If $\alpha-\lambda\alpha-\lambda\omega = 0$, then everything is proven. Assuming $\alpha-\lambda\alpha-\lambda\omega \neq 0$ and using the first equation in (3.7), we write $\mathbf{u}$ as

$$\mathbf{u} = \frac{\omega}{\alpha-\lambda\alpha-\lambda\omega}A^{-1}B\mathbf{v}\,,$$

and substitute into the second equation to obtain

$$\big((1-\lambda)(2-\omega)Q-2\omega C\big)\mathbf{v} = -\frac{2\omega^2\lambda}{\alpha-\lambda\alpha-\lambda\omega}B^{\mathsf{T}}A^{-1}B\mathbf{v}\,.$$

It is easily seen that $\mathbf{v} \neq 0$ so that the term $\mathbf{v}^*Q\mathbf{v}$ is nonzero (since Q is a positive definite matrix), hence on multiplying the above equation by $\mathbf{v}^*/(\mathbf{v}^*Q\mathbf{v})$ from the left we obtain

$$(\alpha-\lambda\alpha-\lambda\omega)\big[(\lambda-1)(2-\omega)+2\omega\eta\big] = 2\omega^2\lambda\gamma\,,$$

or

$$\lambda^2+\left[\frac{2\omega^2\gamma}{(2-\omega)(\alpha+\omega)}-\frac{2\alpha+\omega}{\alpha+\omega}+\frac{2\omega}{2-\omega}\eta\right]\lambda+\left(1-\frac{2\omega}{2-\omega}\eta\right)\left(\frac{\alpha}{\alpha+\omega}\right) = 0\,.$$

This completes the proof. □

Remark 3.1. If in addition to the assumptions of Theorem 3.1 the matrix C is symmetric and positive semidefinite, then $\eta = \mathbf{v}^*C\mathbf{v}/(\mathbf{v}^*Q\mathbf{v}) \geq 0$. For $\eta = 0$ and $\lambda(\alpha + \omega) \neq \alpha$, the eigenvalue λ satisfies

$$\lambda^2 + \left[\frac{2\omega^2\gamma}{(2-\omega)(\alpha+\omega)} - \frac{2\alpha+\omega}{\alpha+\omega}\right]\lambda + \frac{\alpha}{\alpha+\omega} = 0. \tag{3.8}$$

Remark 3.2. For the augmented system of linear equations (1.2), the parameter η is equal to zero; and if $\lambda(\alpha + \omega) \neq \alpha$, then λ satisfies the equation (3.8), consistent with Ref. [20].

Remark 3.3. In the special case $C = \delta Q$, $\delta > 0$, one has $\eta = \delta$ and the representation (3.2) can be written as

$$\begin{pmatrix} A & B \\ -B^{\mathsf{T}} & C \end{pmatrix} = \begin{pmatrix} \alpha A & 0 \\ 0 & \frac{1}{\delta}C \end{pmatrix} - \begin{pmatrix} -A & 0 \\ B^{\mathsf{T}} & \frac{1}{2\delta}C \end{pmatrix} - \begin{pmatrix} \alpha A & -B \\ 0 & (\frac{1}{2\delta}-1)C \end{pmatrix}.$$

To consider the ASOR method for the problem (1.1), we need an auxiliary result.

Lemma 3.1 (cf. Young [24]). *If b and c are real numbers, then the roots of the quadratic equation $x^2 - bx + c = 0$ are less than one in modulus if and only if $|c| < 1$ and $|b| < 1 + c$.*

We now proceed to derive convergence conditions for the ASOR method, on writing

$$\eta_{max} := \max \eta(\mathbf{v}), \quad \gamma_{max} := \max \gamma(\mathbf{v}),$$

where $\eta(\mathbf{v})$ and $\gamma(\mathbf{v})$ are defined in Eqs. (3.6).

Theorem 3.2. *Let $A \in \mathbb{R}^{m \times m}$, $Q \in \mathbb{R}^{n \times n}$ be symmetric positive definite matrices and $B \in \mathbb{R}^{m \times n}$ be a full column rank matrix. If $\lambda(\alpha + \omega) = \alpha$, or if $\lambda(\alpha + \omega) \neq \alpha$ and the parameters η, α and ω satisfy one of the conditions*

a) $\eta = 0$, $0 < \omega < 2$ and $\dfrac{\omega^2 \gamma_{max}}{2 - \omega} < 2\alpha + \omega$

or

b) $\eta > 0$, $0 < \omega < \dfrac{2}{\eta_{max} + 1} < 2$, and also the inequality

$$(\gamma_{max} + \eta_{max} + 1)\omega^2 + 2(\alpha + \eta_{max}\alpha - 1)\omega - 4\alpha < 0 \tag{3.9}$$

holds, then the iteration method (3.3) converges.

Proof. Since $\alpha > 0$ and $\omega > 0$, when $\lambda(\alpha+\omega) = \alpha$, the eigenvalue satisfies the inequality $|\lambda| = \alpha/(\alpha + \omega) < 1$, so the iteration method converges.

For $\lambda(\alpha + \omega) \neq \alpha$ on the other hand, there are two cases to consider — viz. $\eta = 0$ and $\eta > 0$. If $\eta = 0$, the related quadratic equation has the form (3.8), and the proof of the convergence is similar to that of Theorem 1 in Ref. [20]. Now we consider the other case

$\eta > 0$. In order to use Lemma 3.1 the coefficients of the equation (3.5) have to satisfy the inequalities

$$\left| \left(1 - \frac{2\omega}{2-\omega}\eta \right) \left(\frac{\alpha}{\alpha+\omega} \right) \right| < 1 \,, \tag{3.10}$$

$$\left| \frac{2\omega^2\gamma}{(2-\omega)(\alpha+\omega)} - \frac{2\alpha+\omega}{\alpha+\omega} + \frac{2\omega}{2-\omega}\eta \right| < 1 + \left(1 - \frac{2\omega}{2-\omega}\eta \right) \left(\frac{\alpha}{\alpha+\omega} \right). \tag{3.11}$$

If

$$\left| 1 - \frac{2\omega}{2-\omega}\eta \right| < 1, \tag{3.12}$$

then inequality (3.10) is fulfilled because $0 < \alpha/(\alpha+\omega) < 1$. This inequality (3.12) is equivalent to

$$(\eta^2 + \eta)\omega^2 - 2\eta\omega < 0 \,. \tag{3.13}$$

Since $\omega > 0$, $\eta > 0$, the relation

$$\omega < \frac{2}{1+\eta} < 2, \tag{3.14}$$

implies the inequality (3.10). The inequality (3.11) is equivalent to

$$-1 - \left(1 - \frac{2\omega}{2-\omega}\eta \right) \left(\frac{\alpha}{\alpha+\omega} \right) < \frac{2\omega^2\gamma}{(2-\omega)(\alpha+\omega)} - \frac{2\alpha+\omega}{\alpha+\omega} + \frac{2\omega}{2-\omega}\eta$$

$$< 1 + \left(1 - \frac{2\omega}{2-\omega}\eta \right) \left(\frac{\alpha}{\alpha+\omega} \right), \tag{3.15}$$

so we rewrite the left-hand side as

$$\frac{2\omega^2\gamma + 2\omega^2\eta}{(2-\omega)(\alpha+\omega)} > 0$$

to validate the last inequality for all parameters $\omega \in (0,2)$ since γ, α and η are positive. The right-hand side of inequality (3.15) implies

$$\frac{2(\gamma+\eta+1)\omega^2 + 4(\alpha+\eta\alpha-1)\omega - 8\alpha}{(2-\omega)(\alpha+\omega)} < 0 \,,$$

if

$$(\gamma+\eta+1)\omega^2 + 2(\alpha+\eta\alpha-1)\omega - 4\alpha < 0 \,.$$

Consequently, from the condition (3.9) we conclude that the coefficients of the equation (3.5) satisfy all the conditions of Lemma 3.1 and the proof of Theorem 3.1 follows. □

Remark 3.4. Theorem 3.2 shows that if $\eta > 0$, $\omega(1+\eta_{max}) < 2$ and

$$\alpha > \frac{2\omega - (\gamma_{max} + \eta_{max} + 1)\omega^2}{2\omega + 2\eta_{max}\omega - 4} \,,$$

then the iteration method (3.3) converges.

4. Numerical Experiments

The results of numerical experiments conducted on a PC equipped with 2.30 GHz CPU and 2GB RAM in the MATLAB 7.11.0 (R2010b) environment with the machine precision $\epsilon \approx 2.22 \times 10^{-16}$ are now discussed. In the tables below, **IT** means the number of iteration steps, **CPU** the CPU time (seconds), and **RES** the residual defined by

$$\mathbf{RES} := \frac{\sqrt{\|\mathbf{b} - A\mathbf{x}^{(k)} - B\mathbf{y}^{(k)}\|_2^2 + \|\mathbf{q} - B^{\mathsf{T}}\mathbf{x}^{(k)} + C\mathbf{y}^{(k)}\|_2^2}}{\sqrt{\|\mathbf{b}\|_2^2 + \|\mathbf{q}\|_2^2}},$$

where $((\mathbf{x}^{(k)})^{\mathsf{T}}, (\mathbf{y}^{(k)})^{\mathsf{T}})^{\mathsf{T}}$ is the k-th iteration with the the starting point

$$\left((\mathbf{x}^{(0)})^{\mathsf{T}}, (\mathbf{y}^{(0)})^{\mathsf{T}}\right)^{\mathsf{T}} = (0, 0, \cdots, 0)^{\mathsf{T}} \in \mathbb{R}^{m+n}.$$

Example 4.1. We consider the incompressible steady state Stokes problem in the square $D := [-1, 1] \times [-1, 1]$ [12]:

$$\begin{cases} -\nu\Delta\mathbf{u} + \text{grad } \mathbf{p} = f, \\ \text{div } \mathbf{u} = 0, \end{cases} \tag{4.1}$$

where Δ is the Laplace operator, ν denotes the viscosity of the fluid, $\mathbf{p}$ the pressure, and $\mathbf{u}$ the fluid velocity. The boundary conditions assumed are $\mathbf{u}_x = \mathbf{u}_y = 0$ on the lines $x = -1$, $y = -1$, $x = 1$ and $\mathbf{u}_x = 1$, $\mathbf{u}_y = 0$ on $y = 1$.

The problem (4.1) was discretised using uniform square grids and the Incompressible Flow Iterative Solution Software (IFISS) developed by Elman *et al.* [15]. As finite elements, the bilinear constant velocity-pressure $Q_1 - P_0$ pair with $\beta = 0.25$ for local stabilisation was used. The resulting linear system has the form

$$\begin{pmatrix} A & B \\ -B^{\mathsf{T}} & C \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ -\mathbf{q} \end{pmatrix}, \tag{4.2}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the velocity and the pressure on the grid, respectively.

We compare the ASOR, SOR-like [16], and NSOR [18] methods for the problem (1.1). The iteration schemes are stopped when either $RES < 10^{-9}$ or the iteration number 2500 is reached. For all methods, Q is the approximate Schur complement matrix $B^{\mathsf{T}}P^{-1}B + C$ with $P = \text{diag}(A)$. Table 1 contains the optimal parameters obtained in experiments. Table 1 and Fig. 1 show that for the problem (4.1) the ASOR method converges much faster than the two others. We note that the ASOR method is similar to the parameterized inexact Uzawa (PIU) method with the parameters $\omega' = \alpha/(\alpha + \omega)$ and $\tau' = 2\omega/(2 - \omega)$ — cf. Ref. [8]. In our example, for both methods we used the same **IT** but slightly different **CPU**s, so the PIU results are not included in Table 1. Indeed, we recall that Hadjidimos [17] noted the equivalence of optimal relaxed block iterative methods for singular non-symmetric saddle point problems.
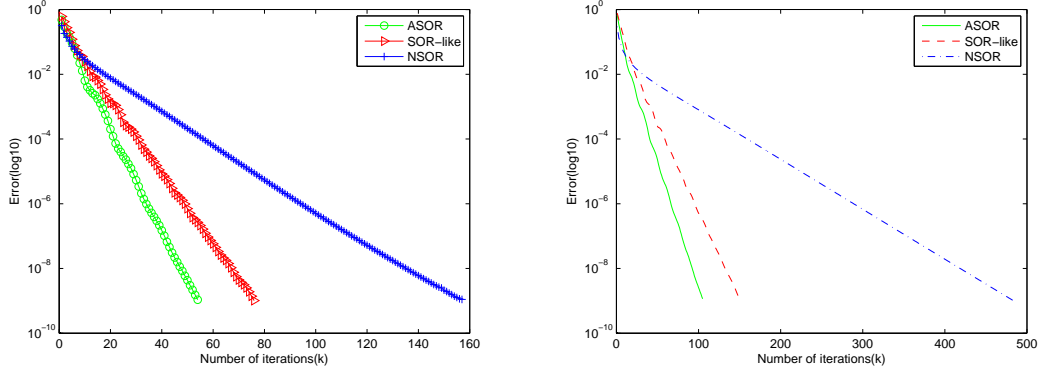
Figure 1: IT for the ASOR, SOR-like and NSOR methods with grids 8×8 and 16×16 in Example 4.1.

Table 1: IT, CPU and RES for the ASOR, SOR-like and NSOR methods in Example 4.1.

Grid:		8×8	16×16	32×32
ASOR	ω_{opt}	0.40	0.23	0.13
	α_{opt}	0.13	0.21	0.30
	IT	27	54	105
	CPU	0.0312	0.236	4.4303
	RES	5.421e-10	7.616e-10	9.413e-10
SOR-like	ω_{opt}	0.65	0.38	0.21
	IT	35	76	151
	CPU	0.0406	0.312	6.424
	RES	4.115e-10	9.844e-10	9.474e-10
NSOR	ω_{opt}	0.69	0.89	0.96
	α_{opt}	1.67	1.37	0.95
	IT	83	157	483
	CPU	0.0696	0.516	15.9308
	RES	6.891e-10	9.932e-10	9.947e-10

Example 4.2. We consider the generalised saddle point problem (1.1) with the following coefficient matrices [18, 25]:

$$A = \begin{pmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{pmatrix} \in \mathbb{R}^{2p^2 \times 2p^2},$$

$$B = \begin{pmatrix} I \otimes F \\ F \otimes I \end{pmatrix} \in \mathbb{R}^{2p^2 \times p^2}, \quad C = I \in \mathbb{R}^{p^2 \times p^2},$$

$$T = \frac{1}{h} \cdot \text{tridiag}\,(-1, 2, -1) \in \mathbb{R}^{p \times p}, \quad F = \frac{1}{h} \cdot \text{tridiag}\,(-1, 1, 0) \in \mathbb{R}^{p \times p},$$

where $h = 1/(p+1)$ denotes the mesh size, $\otimes$ the Kronecker product, and $U = \text{tridiag}\,(a, b, c)$ is the tridiagonal matrix with nonzero entries $u_{i,i-1} = a$, $u_{i,i} = b$, $u_{i,i+1} = c$.

We set $m = 2p^2$, $n = p^2$, so the total number of unknowns is $m + n = 3p^2$. Moreover, the right-hand-side $(\mathbf{b}^{\mathsf{T}}, \mathbf{q}^{\mathsf{T}})^{\mathsf{T}} \in \mathbb{R}^{m+n}$ is the image of the vector $((\mathbf{x}_*)^{\mathsf{T}}, (\mathbf{y}_*)^{\mathsf{T}})^{\mathsf{T}} = (1, 1, \cdots, 1)^{\mathsf{T}} \in$
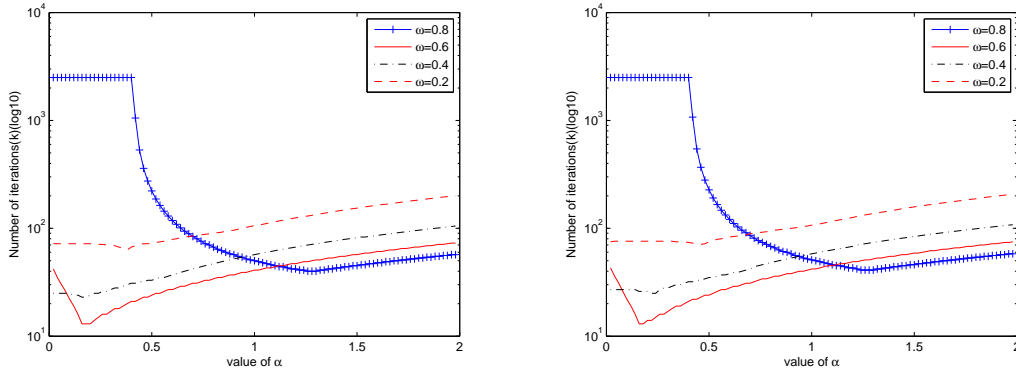
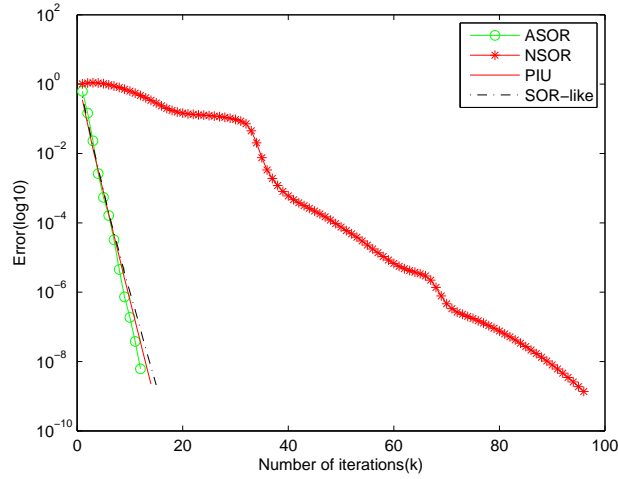Figure 2: IT for the ASOR method with respect to α and ω for $p = 16$ and $p = 32$ in Example 4.2.

Figure 3: Number of iterations for ASOR, PIU, SOR-like and NSOR methods ($p = 16$) in Example 4.2.

$\mathbb{R}^{m+n}$ under the mapping (1.1). The iteration is stopped when either

$$ERR = \frac{\sqrt{\|\mathbf{x}^{(k)} - \mathbf{x}_*\|_2^2 + \|\mathbf{y}^{(k)} - \mathbf{y}_*\|_2^2}}{\sqrt{\|\mathbf{x}^{(0)} - \mathbf{x}_*\|_2^2 + \|\mathbf{y}^{(0)} - \mathbf{y}_*\|_2^2}} < 10^{-9}$$

or the iteration number 2500 is reached.

We compare the ASOR, PIU [8], SOR-like [16], and NSOR [18] methods for the problem (1.1). In this example, we set $Q = C$ for (3.3), similarly to NSOR method in Ref. [18], but in the PIU and SOR-like methods we follow the settings in Ref. [8] and put $P = A$ and $Q = B^{\mathsf{T}}P^{-1}B + C$. For all methods, the experimental optimal parameters α and ω are used. The results presented in Table 2 and Figs. 2 and 3 show the efficiency of our method.

Table 2: IT, CPU and RES for the ASOR, PIU, SOR-like and NSOR methods in Example 4.2.

p		16	24	32	40	48
m		512	1052	2048	3200	4608
n		256	576	1024	1600	2304
$m+n$		768	1728	3072	4800	6912
ASOR	ω_{opt}	0.58	0.58	0.58	0.58	0.58
	α_{opt}	0.14	0.14	0.14	0.14	0.14
	IT	12	12	12	13	13
	CPU	0.0328	0.158	0.495	1.278	2.735
	RES	4.905e-10	5.137e-10	4.833e-10	1.161e-10	1.135e-10
PIU	ω_{opt}	0.88	0.86	0.86	0.58	0.86
	α_{opt}	0.83	0.83	0.83	0.83	0.84
	IT	14	15	16	16	16
	CPU	0.0515	0.256	0.817	2.047	4.313
	RES	5.262e-10	3.718e-10	1.081e-10	1.180e-10	2.688e-10
SOR-like	ω_{opt}	0.85	0.84	0.84	0.84	0.84
	IT	15	16	16	16	16
	CPU	0.0452	0.267	0.819	2.050	4.322
	RES	6.199e-10	1.727e-10	2.915e-10	2.782e-10	2.688e-10
NSOR	ω_{opt}	1.58	1.64	1.72	1.78	1.80
	α_{opt}	0.24	0.19	0.16	0.13	0.10
	IT	96	150	178	222	275
	CPU	0.0515	0.429	1.420	4.065	9.850
	RES	4.307e-10	3.244e-10	1.254e-10	2.487e-10	1.668e-10

Conclusion

We have applied the ASOR method [20] to the generalised saddle point problem (1.1) with a symmetric positive semidefinite matrix, and provided sufficient conditions for its convergence.

Acknowledgments

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