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Critical Angles of the Nyström Method for Double Layer Potential Equation

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Abstract. The Nyström method for double layer potential equations on simple closed piecewise smooth contours is stable if and only if certain operators associated with the opening angles of the corners are invertible. Numerical experiments show that there are opening angles which cause the instability of the method.

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1. Introduction

Boundary integral equations are widely used in approximate solution of partial differential equations — e.g. the Dirichlet problem for the Laplace equation

$$\begin{aligned}\Delta u(x,y) &= 0, \quad (x,y)\in D\\ u(x,y) &= f(x,y), \quad (x,y)\in\Gamma\end{aligned} \tag{1.1}$$

in a simply connected domain $D\subset\mathbb{R}^2$ can be reduced to a double layer potential equation on the boundary Γ of D.

We consider the double layer potential equation

$$(Ax)(t) = x(t) + \frac{1}{\pi}\int_\Gamma x(\tau)\frac{d}{dn_\tau}\log|t-\tau|\,d\Gamma_\tau + (Tx)(t) = f(t), \quad t\in\Gamma. \tag{1.2}$$

where n_τ refers to the outer normal to Γ at the point $\tau\in\Gamma$ and T is a compact operator. The form of T depends on the boundary value problem studied. In particular, if the following integral representation

$$u(z) := \frac{1}{\pi}\int_\Gamma x(\tau)\frac{d}{dn_\tau}\log|z-\tau|\,d\Gamma_\tau, \quad z\in D,$$

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of the solutions of (1.1) is used, then T is the zero operator [2]. The operator A in (1.2) has been thoroughly studied for various contours and spaces — cf. Refs. [8,22,25,27]. For smooth curves Γ, the double layer potential operator

$$V_\Gamma x(t) := \frac{1}{\pi}\int_\Gamma x(\tau)\frac{d}{dn_\tau}\log|t-\tau|\,d\Gamma_\tau \tag{1.3}$$

acting in an L^p-space is compact. This property of V_Γ substantially simplifies the stability analysis of approximation methods. In fact, for smooth curves, the invertibility of A and a good convergence of approximation operators to A ensure the stability of the method. For curves with corners, the operator (1.3) is not compact and stability depends on additional parameters — cf. Refs. [2, 12, 13, 18, 21, 22]. Here we consider the Nyström discretisation based on Gauss-Legendre quadrature formula. This method is easily implemented and demonstrates a good convergence even for contours with a large set of corners — cf. Refs. [4, 5, 20]. Its stability for the Sherman-Lauricella and Muskhelishvili equations has been studied earlier [9–11, 15]. For the double layer potential equation, the stability is proven for sufficiently nice curves — polygons or curves coinciding with polygons in neighbourhoods of corners, or/and for sufficiently nice double layer potential operators — perturbations of the identity by compact and small norm operators [18, 21, 22]. Besides, so far only sufficient stability conditions are available. Our goal here is to establish necessary and sufficient stability conditions of this method for general simple piecewise smooth curves and to provide a procedure for their verification. It turns out that there exist angles which cause instability. Such angles are called critical and if Γ has a critical angle, the Nyström method is not stable regardless of the shape of the curve. In contrast, spline Galerkin methods do not have any critical angles [15, 17] but the complexity of the implementation and computational cost are significantly higher than for Nyström methods. Therefore, in practical computations Nyström type methods are preferable and there are modifications designed to handle the instability induced by non-smooth boundaries [7, 19]. On the other hand, in the interval $[0.1\pi, 1.9\pi]$, the Nyström method for the double layer potential equation has only four critical angles. It is possible that some or all of them are irrational, so we may never encounter the unstable situation during implementation of the method — cf. Remark 3.1 and Fig. 3 below.

Let us make a few technical remarks. We identify every point (x, y) of $\mathbb{R}^2$ with the point $z = x + iy$ in the complex plane $\mathbb{C}$. By S_Γ we denote the Cauchy singular integral operator on Γ,

$$(S_\Gamma x)(t) := \frac{1}{\pi i}\int_\Gamma \frac{x(\tau)\,d\tau}{\tau - t}$$

and by M the operator of complex conjugation, $M\varphi(t) := \overline{\varphi(t)}$. It is known [23] that the double layer potential operator V_Γ can be represented in the form

$$V_\Gamma = \frac{1}{2}(S_\Gamma + MS_\Gamma M),$$